Lanchester's Equations in Three Dimensions

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Abstract

This paper generalizes Lanchester's equations of warfare to partial differential equations involving time and two spatial variables. Unlike in Lanchester's original ordinary differential equations, the distribution of armies over the battlefield must be considered. Four different modes of attack are introduced, generalizing Lanchester's equations for area fire and for direct fire. The effect of the distribution of forces and their movement on the outcome is considered, and numerical simulations given.
Keywords: Lanchester's Equations, area fire, direct fire, reaction-diffusion equations, battlefield simulation

1 Introduction

In 1914, Lanchester formulated his well-known equations of warfare ([1]). They were discovered concurrently and independently by Osipov ([2]). Consider a Red army and a Blue army, with populations $R = R(t)$ and $B = B(t)$ varying in time. Lanchester's Equations for Area (unaimed) Fire are:

$$\frac{dR}{dt} = -\alpha BR, \qquad \frac{dB}{dt} = -\beta BR, \quad \alpha, \beta > 0. \tag{1.1}$$

Attrition is proportional to product of army populations. In Lanchester's Equations for Direct (aimed) Fire,

$$\frac{dR}{dt} = -\alpha B, \qquad \frac{dB}{dt} = -\beta R, \quad \alpha, \beta > 0. \tag{1.2}$$

Attrition is proportional to the attacking army's population. Lanchester himself considered the latter set of equations to be more descriptive of modern warfare

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([3]). The equations can be solved analytically. For the Area Fire equations, we obtain Lanchester's linear law:

$$\frac{R_0 - R(t)}{B_0 - B(t)} = \frac{\beta}{\alpha}, \tag{1.3}$$

for all $t \geq 0$, where $R_0 = R(0)$ and $B_0 = B(0)$. For the Direct Fire equations, we obtain Lanchester's quadratic law:

$$\frac{R_0^2 - R(t)^2}{B_0^2 - B(t)^2} = \frac{\beta}{\alpha}, \tag{1.4}$$

Lanchester's Equations have been used extensively to model combat ([4], [5], [6], [7]), as well as in behavioral ecology ([8]) and consumer marketing ([9], [10]). The Web page www.lanchester.com contains links to papers and books about the application of Lanchester's equations to marketing strategy At the same time, the relevance of these equations to modern warfare is the subject of much controversy ([11]).

One reason for this is that Lanchester's Equations do not contain any information about the spatial distribution of armies or their movement. Here we propose a partial differential equation (PDE) model that accounts for the temporal and spatial distribution of armies. Four different forms of attack, based on (1.1) or (1.2), are examined. We examine the behavior of solutions of the resulting PDE. Given two armies with similar capabilities, we attempt to determine, for each attack mode, what strategy an army should follow (what its initial distribution should be and how it moves in the plane) to maximize enemy losses and minimize its own. Outcomes of numerical experiments of battles are given. All the numerical experiments were conducted on a PC running Windows 98, 2d edition, with a Pentium III processor running at 750 MHz. The calculations for Sections 2, 3, 4, and 5 were done in Java 1.5.1, and the calculations for Sections 6 and 7 were done in Maple 9.

The idea of using PDE for modeling warfare is not new: see [12] for a general discussion, [13] for some numerical results in the case of one spatial dimension, [14] for applying reaction-diffusion equations to model troop movements, [15] for analyzing troop maneuvers in the plane, [16] for rigorous existance results for PDE versions of Lanchester's equations, and [17] and [18] for some rigorous mathematical results using tools of control theory and differential games. However, we believe this paper contains some new ideas, particularly the surrounding direct fire idea of Section 5, that are ripe for more study and which may prove useful in applications.

2 Local Direct Fire

The most obvious way to extend Lanchester's equations to the time-space domain is to imagine each army as a continuous fluid, with nonnegative concentrations $r = r(x, y, t)$ and $b = b(x, y, t)$. Armies move about the plane, and inflict attrition on each other, locally or at a distance, following various rules. Let $\mathbf{v}_r = \mathbf{v}_r(x, y, t)$ be a vector field describing velocity (direction and speed) of the Red army at point (x, y) and time t. Let $\mathbf{v}_r = \mathbf{v}_r(x, y, t)$ be a vector field describing velocity (direction and speed) of the Blue army at point (x, y) and time t. Let r and b satisfy

$$\frac{\partial r}{\partial t} = -\nabla r \cdot \mathbf{v}_r - I_r, \quad \frac{\partial b}{\partial t} = -\nabla b \cdot \mathbf{v}_b - I_b, \tag{2.1}$$

where $I_r = I_r(x, y, t, r, b)$ and $I_b = I_b(x, y, t, r, b)$ are the attrition rates for Red and Blue at location (x, y) and time t.

In this paper we propose four different forms for the attrition terms I_r , I_b . Each form is the subject of one section. These are based on Lanchester's rules for direct or area fire.

The simplest way to extend Lanchester's Equations for Direct Fire to the PDE setting would be to make $I_r(x, y, t)$ proportional to $b(x, y, t)$ and $I_b(x, y, t)$ proportional to $r(x, y, t)$ in (2.1), obtaining

$$\begin{aligned} \frac{\partial r}{\partial t} &= -\nabla r \cdot \mathbf{v}_r - \alpha b(x, y, t), \\ \frac{\partial b}{\partial t} &= -\nabla b \cdot \mathbf{v}_b - \beta r(x, y, t) \end{aligned} \tag{2.2}$$

for some $\alpha, \beta > 0$. This has the drawback that $r(x, y, t)$ and $b(x, y, t)$ may become negative at certain points (x, y) in the plane for arbitrarily small values of t! We can remedy this flaw by defining

$$\begin{aligned} I_r(x, y, t) &= \alpha b(x, y, t) \, \frac{r(x, y, t)}{\epsilon + r(x, y, t)}, \\ I_b(x, y, t) &= \beta r(x, y, t) \, \frac{b(x, y, t)}{\epsilon + b(x, y, t)}, \end{aligned} \tag{2.3}$$

where ϵ is a small positive constant. Then, except for small $r(x, y, t)$, $I_r(x, y, t) \approx \alpha b(x, y, t)$, and except for small $b(x, y, t)$, $I_b(x, y, t) \approx \beta r(x, y, t)$. Just as with direct fire with the original Lanchester equations, an army can win by concentrating its forces. For example, consider the equations

$$\begin{aligned} \frac{\partial r}{\partial t} &= -\nabla r \cdot \mathbf{v}_r - 10\, b(x, y, t) \frac{r(x, y, t)}{0.01 + r(x, y, t)}, \\ \frac{\partial b}{\partial t} &= -10\, r(x, y, t) \frac{b(x, y, t)}{0.01 + b(x, y, t)}, \end{aligned} \tag{2.4}$$

with initial conditions

$$r(x, y, 0) = \begin{cases} 2; & 0.1 \le x \le 0.2, \ \ 0.3 \le y \le 0.4; \\ 0; & \text{otherwise}, \end{cases} \tag{2.5}$$

$$b(x, y, 0) = \begin{cases} 0.5; & 0.21 \le x \le 0.61, \ 0.3 \le y \le 0.4; \\ 0; & \text{otherwise}, \end{cases} \tag{2.6}$$

$\mathbf{v}_r(x,y,t) =< 1-x, 0 >$, and $\mathbf{v}_b(x,y,t) =< 0, 0 >$. The Blue army is immobile. $R_0 = 0.02 = B_0$, but the Red army is more tightly concentrated. The Red army sweeps across the Blue army, inflicting more attrition than Blue does. Using the same numerical method as in Section 2, the army populations in the solution of (2.1) behave as shown in Figure 1. Figure 2 shows the armies at times $t = 0, 0.3$, and 0.6.

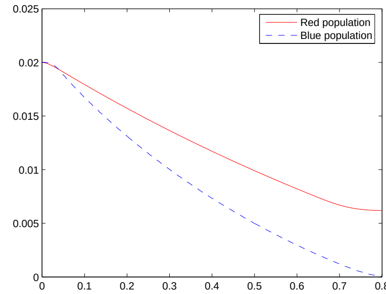
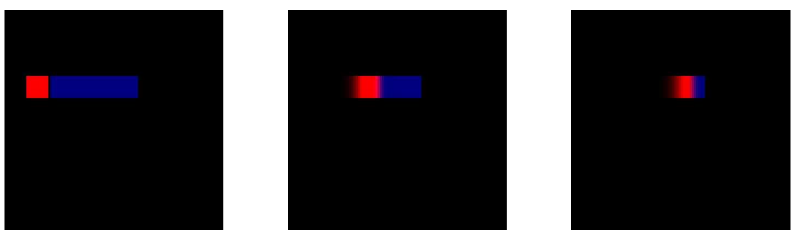


Figure 1: Local Direct Fire example: populations as functions of time



$t = 0$ $t = 0.3$ $t = 0.6$

Figure 2: Local Direct Fire example: Distributions at $t = 0, 0.3, 0.6$

The simulation used a timestep of $\Delta t = 0.001$, and the experiment took 44 sec.

Blue fares better if he adopts an active defense, concentrating his forces even more than Red's while waiting for Red to arrive. Define

$$\mathbf{v}_b(x,y,t) =< 3.0(0.6-x), 0 >, \tag{2.7}$$

and alter (2.4) as follows:

$$\begin{aligned} \frac{\partial r}{\partial t} &= -\nabla r \cdot \mathbf{v}_r - 10\, b(x,y,t) \frac{r(x,y,t)}{0.01 + r(x,y,t)}, \\ \frac{\partial b}{\partial t} &= -\nabla b \cdot \mathbf{v}_b - 10\, r(x,y,t) \frac{b(x,y,t)}{0.01 + b(x,y,t)}, \end{aligned} \tag{2.8}$$

with the same initial conditions $r(x, y, 0)$ and $b(x, y, 0)$. This time, when the armies meet, Blue is more concentrated than Red, and the army populations behave as in Figure 5. Blue is victorious. Figure 4 shows the armies at times

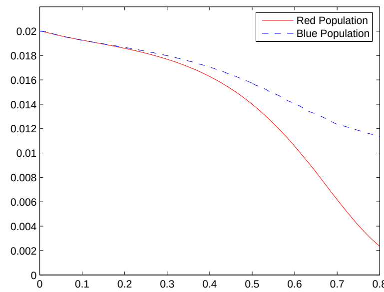


Figure 3: Local Direct Fire example: Blue concentrates to defeat Red

$t = 0.3$ and 0.8. The simulation used a timestep of $\Delta t = 0.001$, and took 18 sec. to run.

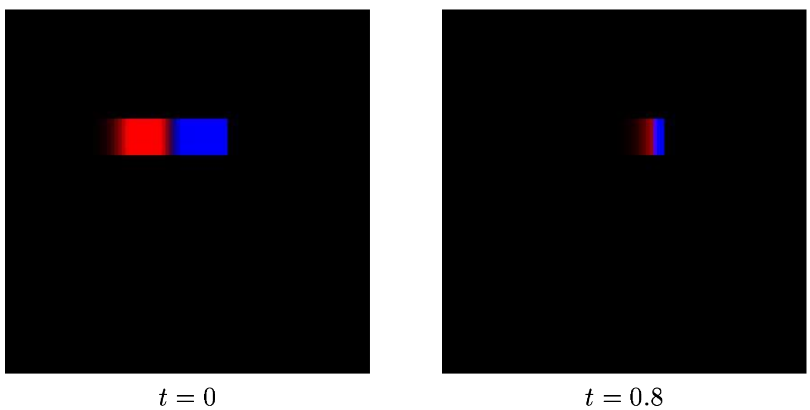


Figure 4: Local Direct Fire example: $t = 0.3$ and $t = 0.8$

In conclusion, using Direct Area Fire, the army which is more concentrated in space will defeat an equally large enemy.

3 Nonlocal Area Fire

Our next model is for attrition inflicted over a distance. Assume that each Red soldier has an identical weapon, and there exists a function $\varphi_r : \mathbb{R}^2 \to \mathbb{R}^+$ that describes the effectiveness of the weapon: if the weapon is deployed by

Red forces at the point (x, y), then the rate of attrition suffered by Blue at the point (x', y') is proportional to $\varphi_r(x - x', y - y')r(x, y)b(x, y)$. For mathematical simplicity, we will assume that φ_r is circularly symmetric, which is appropriate if the weapon can be aimed in any direction equally easily. Assume $\varphi_r \geq 0$ and $\iint_{\mathbb{R}^2} \varphi_r dx\, dy < \infty$. Let the attrition suffered by Blue at (x, y) be the summation, or integral, of the attrition inflicted by nearby Red forces. Then I_b in (2.1) is given by

$$I_b(x, y, t) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_r(x - x', y - y')r(x', y', t)dx'\, dy' \right) b(x, y, t). \quad (3.1)$$

The double integral is the convolution of φ_r with r. $I_r(x, y, t)$ is defined similarly.

Many choices for φ_r are possible, but we choose the bivariate Gaussian $\varphi_r(x, y) = (A_r/(2\pi\sigma_r^2)) \exp(-(x^2 + y^2)/(2\sigma_r^2))$, because indirect area fire's range probably error is based on a bivariate Gaussian. A_r and σ_r are positive parameters. Likewise, $\varphi_b(x, y) = (A_b/(2\pi\sigma_b^2)) \exp(-(x^2 + y^2)/(2\sigma_b^2))$. If $\sigma_r = \sigma_b$, then it is intuitively clear that Red's losses are proportional to Blue's with ratio A_b/A_r , and Lanchester's linear law (1.3) holds. For example, define

$$r(x, y, 0) = \begin{cases} 1; & 0.1 \leq x \leq 0.3,\, 0.1 \leq y \leq 0.3, \\ 0; & \text{otherwise}, \end{cases}, \quad (3.2)$$

$$b(x, y, 0) = \begin{cases} 1; & 0.2 \leq x \leq 0.4,\, 0.5 \leq y \leq 0.9, \\ 0; & \text{otherwise}, \end{cases}, \quad (3.3)$$

$\sigma_r = \sigma_b = 0.1$, $A_r = 20$, $A_b = 40$, and define φ_r, φ_b, I_r and I_b as described above and in (3.1). Let $\mathbf{v}_r(x, y, t)$ be the vector from (x, y) to the centroid of b, and $\mathbf{v}_b(x, y, t)$ be the vector from (x, y) to the centroid of r. $R_0 = 0.04$, $B_0 = 0.08$, and the army populations behave as in Table 1.

t	$R(t)$	$B(t)$	$(R_0 - R(t))/(B_0 - B(t))$
0.1	0.03959959	0.07979955	0.5004168076
0.2	0.036766432	0.07838142	0.5033595005
0.3	0.030219099	0.0751014	0.5101139677
0.4	0.021536902	0.021536902	0.4861670373

Table 1: Nonlocal area fire with equal sigma's

The values for [Red losses]/[Blue losses] are all very close to the theoretical value of $20/40 = 0.5$. Figure 5 shows the populations of the armies as a function of time. Figure 6 shows the armies at times $t = 0$ and 0.4. The simulation used a timestep of $\Delta t = 0.001$, and took 10 min. 7 sec. to run.

In a sense, Red and Blue have equal strength if $A_r = A_b$. These parameters tell how much attrition a soldier can inflict who is completely surrounded by

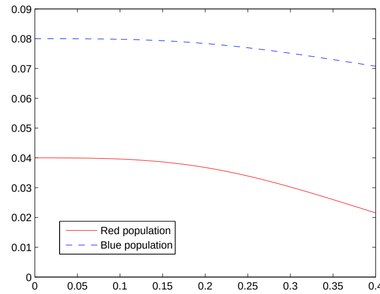
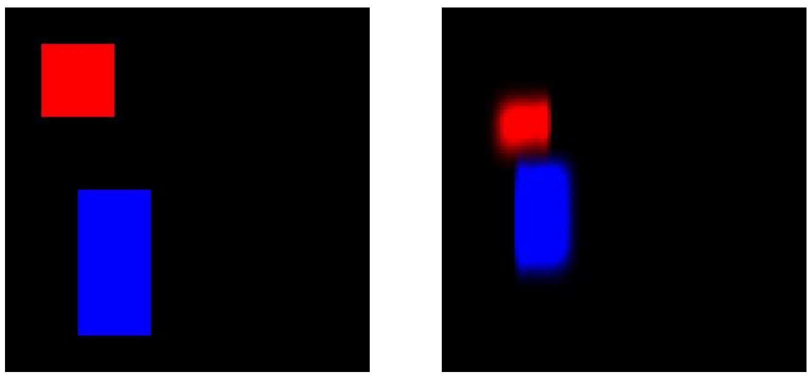


Figure 5: Nonlocal Area Fire: Equal Sigma's



$t = 0$ $\qquad$ $t = 0.4$

Figure 6: Nonlocal Area Fire example: $t = 0$ and $t = 0.4$

enemy. For example, suppose we have an infinite battlefield, with $r(x, y, 0) = r_0$ and $b(x, y, 0) = b_0$ for all (x, y). Then if $A_r = A_b$, $r_0 - r(x, y, t) = b_0 - b(x, y, t)$ for all (x, y) and t. In general, however, if $\sigma_r \neq \sigma_b$, then the outcome of a battle depends on the initial distribution of forces and their movement.

For example, let $r(x, y, 0) = 1$ on the square $[0.1, 0.3] \times [0.1, 0.3]$, 0 elsewhere and let $b(x, y, 0) = 1$ on the square $[0.1, 0.3] \times [0.3, 0.5]$, 0 elsewhere. Define

$$\varphi_r(x, y) = 20/(2\pi(0.1)^2)\exp(-(x^2 + y^2)/(2(0.1)^2)), \tag{3.4}$$
$$\varphi_b(x, y) = 20/(2\pi(0.05)^2)\exp(-(x^2 + y^2)/(2(0.05)^2)),$$

and let I_r and I_b be defined as in (3.1). Let the armies be motionless, with $\mathbf{v}_r = \mathbf{v}_b =< 0, 0 >$. Figure 7 shows the populations of the armies as a function of time, and Figure 8 shows the armies at $t = 0$ and $t = 8$. With its greater value of σ, Red is able to reach farther into Blue territory. The simulation used a timestep of $\Delta t = 0.005$, and took 4 min. 47 sec. to run.

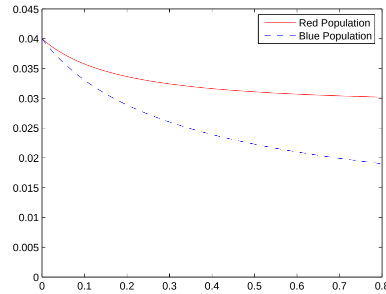
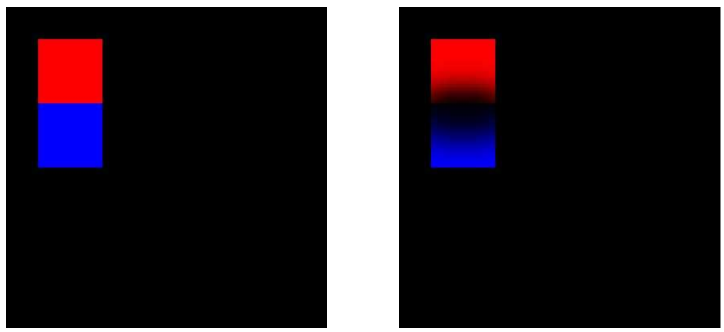


Figure 7: Nonlocal Area Fire: Red reaches into Blue territory



$t = 0$ $t = 0.8$

Figure 8: Nonlocal Area Fire example: $t = 0$ and $t = 8$

Next, let $\varphi_r, \varphi_b, I_r$, and I_b be as above with $\mathbf{v}_r = \mathbf{v}_b =< 0, 0 >$ again. But this time let $r(x, y, 0) = 0.25$ on the square $[0.1, 0.3] \times [0.1, 0.3]$, 0 elsewhere, and let $b(x, y, 0) = 1$ on the rectangle $[0.1, 0.3] \times [0.1, 0.15]$, 0 elsewhere. The army populations remain approximately equal, as shown in Figure 9. The simulation used a timestep of $\Delta t = 0.005$, and took 4 min. 43 sec. to run.

Using Nonlocal Area Fire, a weapon's effectiveness is given by a circularly symmetric bivariate Gaussian. If the associated standard deviation is the same for two opposing armies, then the attrition of the armies follows the well-known Lanchester's linear law, regardless of the spatial distribution or movement of the armies. If the associated standard deviations are different and armies are separated by a distance, then the army whose attack has the larger standard deviation may have the advantage.

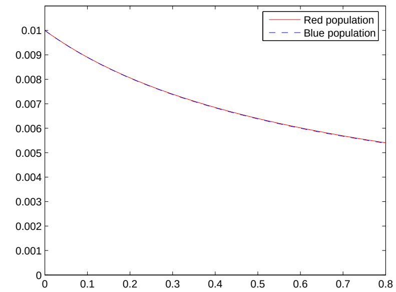


Figure 9: Nonlocal Area Fire example

4 Nonlocal Direct Fire: Front Model

In the next attack model, each soldier fires only at those targets that are reachable to him, and chooses from among these targets in proportion to their attainability. The attrition suffered by an army at a point is proportional to the sum of the attrition inflicted by nearby enemy soldiers.

Let the relative attainability function $\varphi_r \equiv \varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonnegative, circularly symmetric function, with $\varphi(\rho, 0) \to 0$ as $\rho \to \infty$. Define

$$V_{r,b}(x, y) = \iint_{\mathbb{R}^2} \varphi(x' - x, y' - y) b(x', y')\, dx'\, dy'. \tag{4.1}$$

We omit the time t for clarity's sake. Let $A_r > 0$ be a constant indicating the strength of Red's attack. Define $I_{r,b}(x, y, x_2, y_2)$ to be the rate of attrition inflicted by Red forces at (x, y) upon Blue forces at (x_2, y_2) to be

$$I_{r,b}(x, y, x_2, y_2) = A_r r(x, y) \frac{\varphi(x_2 - x, y_2 - y) b(x_2, y_2)}{V_{r,b}(x, y)} \tag{4.2}$$

Note how Red's firepower is spread among available targets. Also note that

$$\iint_{\mathbb{R}^2} I_{r,b}(x, y, x_2, y_2) dx_2 dy_2 = A_r r(x, y). \tag{4.3}$$

$I_{r,b}$ is homogeneous of degree 0 in φ. That is, replacing φ by $\lambda\varphi$ for any $\lambda > 0$ would not change $I_{r,b}$. Finally, define I_b in (2.1) by

$$I_b(x, y) = \iint_{\mathbb{R}^2} I_{r,b}(x', y', x, 0)\, dx'\, dy'. \tag{4.4}$$

The multiple convolutions required to implement this model made the computations prohibitively slow for the general problem with two spatial dimensions. We will assume the armies satisfy a spatial symmetry that effectively reduces the number of spatial dimensions to one. There are essentially two such symmetries: the armies can be radially symmetric about the same center, or the armies can be infinite in extent, with their densities a function of the x-coordinate only. We make the latter assumption. Now $r(x, y, t) = r(x, t)$ and $b(x, y, t) = b(x, t)$. The armies have infinite populations, so we measure their size by $R(t) = \int_{\mathbb{R}} r(x, t)\, dx$ and $B(t) = \int_{\mathbb{R}} b(x, t)\, dx$, $R_0 = R(0)$, $B_0 = B(0)$. Since φ is circularly symmetric, we may define $\varphi(\rho) = \varphi(\rho, 0)$. Then (omitting t for clarity)

$$V_{r,b}(x) = \iint_{\mathbb{R}^2} \varphi(\sqrt{(x' - x)^2 + y'^2})b(x')\, dy'\, dx', \tag{4.5}$$

$$I_{r,b}(x, y, x_2, y_2) = A_r \frac{\varphi(x_2 - x, y_2 - y)b(x_2)}{V_{r,b}(x)}, \tag{4.6}$$

and

$$\begin{aligned} I_b(x) \equiv I_b(x, 0) &= \iint_{\mathbb{R}^2} I_{r,b}(x', y', x, 0)\, dx'\, dy' = \\ &= A_r b(x) \int_{x'=-\infty}^{\infty} \frac{r(x')}{V_{r,b}(x')} \int_{y'=-\infty'}^{\infty} \varphi(x - x', y')dy'\, dx'. \end{aligned} \tag{4.7}$$

$V_{b,r}$, $I_{b,r}$, and I_r are defined by switching the roles of Red and Blue. Our PDE now have the form

$$\begin{aligned} \frac{\partial r}{\partial t} &= -v_r \frac{\partial r}{\partial x} - I_r, \\ \frac{\partial b}{\partial t} &= -v_b \frac{\partial b}{\partial x} - I_b. \end{aligned} \tag{4.8}$$

For example, set $A_r = A_b = 1$ and

$$\varphi_r(x, y) = \varphi_b(x, y) = \exp(-\frac{1}{2}(x^2 + y^2)/0.1^2)/((0.1)^2(2\pi)). \tag{4.9}$$

Since $\varphi_r = \varphi_b$ equals a function of x times a function of y, we obtain

$$\begin{aligned} V_{r,b}(x) &= \tfrac{1}{(0.1)^2(2\pi)} \iint_{\mathbb{R}^2} b(x') \exp(-\tfrac{(x-x')^2+y'^2}{2\cdot 0.1^2})dx'dy' = \\ &= \tfrac{1}{(0.1)\sqrt{2\pi}} \int_{x'=-\infty}^{\infty} \exp(-\tfrac{(x-x')^2}{2\cdot 0.1^2})b(x')dx', \end{aligned} \tag{4.10}$$

$$\iint_{\mathbb{R}^2} \varphi(x - x', y')\, dy' = \tag{4.11}$$

$$= \frac{1}{(0.1)^2(2\pi)} \int_{y'=-\infty}^{\infty} \exp(-\frac{(x - x')^2}{2 \cdot 0.1^2} \exp(-\frac{y'^2}{2 \cdot 0.1^2})\, dy' = \tag{4.12}$$

$$= \frac{1}{(0.1)\sqrt{2\pi}} \exp(-\frac{(x - x')^2}{2 \cdot 0.1^2}),$$

and

$$I_b(x) = b(x) \int_{x'=-\infty}^{\infty} \frac{r(x') \exp(-(x-x')^2/(2 \cdot 0.1^2))}{\int_{\bar{x}=-\infty}^{\infty} \exp(-(x'-\bar{x})^2/(2 \cdot 0.1^2)) b(\bar{x}) \, d\bar{x}} \, dx'. \tag{4.13}$$

Note that I_b is homogeneous of degree 1 in r: replacing r by λr multiplies I_b by λ. I_b is homogeneous of degree 0 in b: multiplying b by $\lambda > 0$ does not change I_b.

Likewise,

$$I_r(x) = r(x) \int_{x'=-\infty}^{\infty} \frac{b(x') \exp(-(x-x')^2/(2 \cdot 0.1^2))}{\int_{\bar{x}=-\infty}^{\infty} \exp(-(x'-\bar{x})^2/(2 \cdot 0.1^2)) r(\bar{x}) \, d\bar{x}} \, dx'. \tag{4.14}$$

Let us take $r(x, 0) = 8$ for $0.0 \leq x \leq 0.1$, 0 otherwise, $b(x) = 1$ for $0.1 \leq x \leq 0.8$, 0 otherwise, and $v_r = v_b = 0$ (motionless armies). Then $R_0 = B_0 = 0.8$, and populations of the armies in the solution of (4.8) behave as in Figure 10. Figure 11 shows the initial and time $t = 1$ states of the armies. This experiment

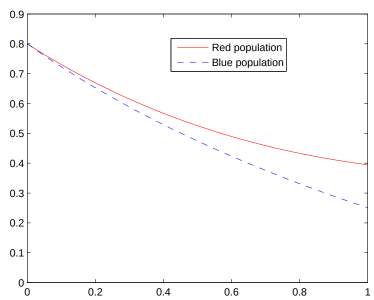


Figure 10: Nonlocal Direct Fire: Concentrated Red defeats Blue

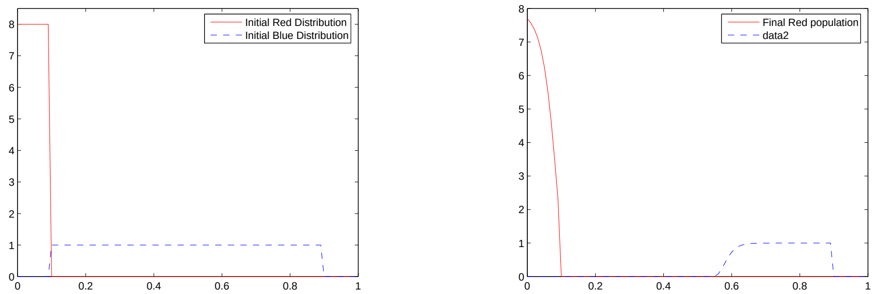
used a timestep of $\Delta t = 0.1$, and ran for 118 sec.

Next, we vary the width of the initial distribution of the Blue army and see how it affects the outcome of the battle. For $width = 0.05, 0.06, \ldots, 0.9$ we let $r(x, y, 0)$ be as above but define $b(x, y, 0)$ by $b(x, y, 0) = 0.8/width$ for $x \in [0.1, 0.1 + width]$, 0 otherwise. Figure 12 shows $R(1)$ and $B(1)$ as functions of $width$. For small widths, the battle is equal, but for large widths, Red has more forces close to Blue than Blue has close to Red. The 86 simulations performed to produce Figure 12 required a total running time of 3 hr. 7 min.

As with local direct fire, the army that concentrates more closely in space has the advantage.

5 Surrounding Direct Fire, Front Model

Our final mode of attack is similar to Nonlocal Direct Fire, except that the attrition suffered by an army is not simply a sum or integral of the attrition



$t = 0$ $\qquad$ $t = 1$

Figure 11: Nonlocal Direct Fire: Initial and Final distributions

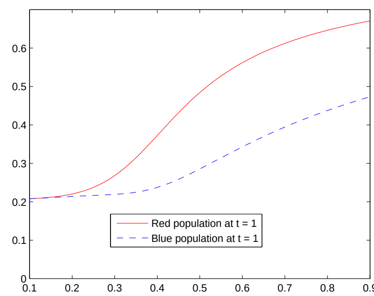


Figure 12: Nonlocal Direct Fire: Concentrated Red defeats Blue

inflicted by nearby enemy, but is computed by a formula that rewards an army for surrounding its foe, or merely for spreading out in the x-direction. Let $V_{r,b}(x, y)$ and $I_{r,b}(x, y, x_2, y_2)$ be as in (4.1),(4.2), and define

$$I_b(x, y) = (\iint_{\mathbb{R}^2} I_{r,b}(x', y', x, y)^{\frac{1}{p}}\, dx'\, dy')^p \tag{5.1}$$

where $p \equiv p_r > 1$ is a parameter describing the Red army. The variable t is again omitted for clarity in what follows. I_b is nonlinear but homogeneous in r. Multiplying r by $\lambda > 0$ multiplies I_b by λ.

Again, the calculations on a two-dimensional battlefield involved an intractible amount of computations, so we assume that r and b depend only on x and t. Now

$$I_b(x) = A_r b(x)(\iint_{\mathbb{R}^2} r(x')^{\frac{1}{p}} \frac{\varphi_r(x' - x, y')^{\frac{1}{p}}}{V_{r,b}(x')^{\frac{1}{p}}}\, dy' dx')^p. \tag{5.2}$$

and likewise

$$I_r(x) = A_b r(x) \left(\iint_{\mathbb{R}^2} b(x')^{\frac{1}{p}} \frac{\varphi_b(x'-x,y')^{\frac{1}{p}}}{V_{b,r}(x')^{\frac{1}{p}}} \, dy' dx' \right)^p. \tag{5.3}$$

Let us examine some experiments in which

$$\varphi_r(x,y) = \varphi_b(x,y) = \exp\left(-\frac{(x^2+y^2)}{2\cdot 0.1^2}\right)/((0.1)^2(2\pi)), \tag{5.4}$$

$A_r = A_b = 0.5$, and $p_r = p_b = 1.2$. Now

$$I_r(x) = \frac{0.1^{0.2}(1.2\pi)^{0.6}}{2\sqrt{2\pi}} b(x) \left(\int_{\mathbb{R}} \frac{\exp(-(x'-x)^2/(2.4\cdot 0.1^2)) r(x')^{\frac{1}{1.2}}}{\int_{\mathbb{R}} \exp(-(x'-\bar{x})^2/(2\cdot 0.1)^2) b(\bar{x}) \, d\bar{x}^{\frac{1}{1.2}}} \right)^{1.2}, \tag{5.5}$$

and

$$I_b(x) = \frac{0.1^{0.2}(1.2\pi)^{0.6}}{2\sqrt{2\pi}} r(x) \left(\int_{\mathbb{R}} \frac{\exp(-(x'-x)^2/(2.4\cdot 0.1^2)) b(x')^{\frac{1}{1.2}}}{\int_{\mathbb{R}} \exp(-(x'-\bar{x})^2/(2\cdot 0.1)^2) r(\bar{x}) \, d\bar{x}^{\frac{1}{1.2}}} \right)^{1.2}. \tag{5.6}$$

Let us take $r(x,0) = 8$ for $0.0 \le x \le 0.1$, 0 otherwise, $b(x) = 4$ for $0.1 \le x \le 0.3$, 0 otherwise, and $v_r = v_b = 0$ (motionless armies). Then $R_0 = B_0 = 0.8$, and populations of the armies behave as in Figure 13. Figure 14 shows the initial and final states of the armies. Blue wins because Blue's forces are more spread out than Red's. Using a timestep of $\Delta t = 0.01$, the experiment required 21

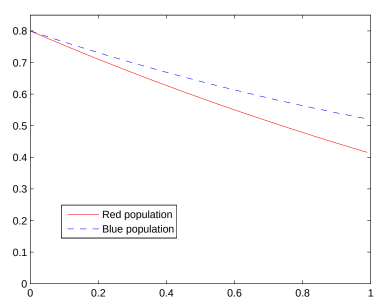
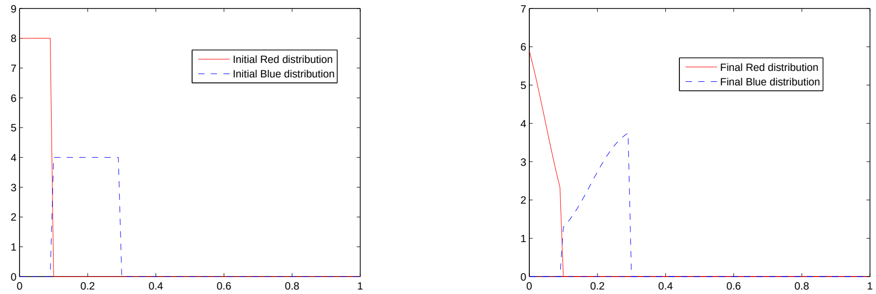


Figure 13: Surrounding Direct Fire: Populations as functions of time

min.

Next, we vary the width of the initial distribution of the Blue army and see how it affects the outcome of the battle. For $width = 0.05, 0.06, \ldots, 0.9$ we let $r(x,y,0)$ be as above but define $b(x,y,0)$ by $b(x,y,0) = 0.8/width$ for $x \in [0.1, 0.1+width]$, 0 otherwise. Figure 15 shows $R(1)$ and $B(1)$ as functions of $width$. There is a tradeoff between concentrating in one place and spreading



$t = 0$ $t = 1$

Figure 14: Surrounding Direct Fire: Initial and Final distributions

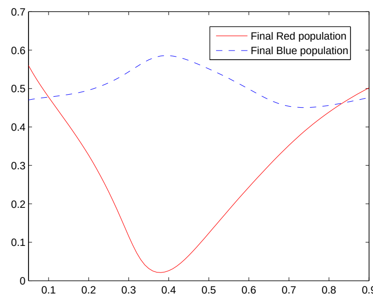


Figure 15: Surrounding Direct Fire: Populations as functions of starting widths

in the x-direction. The 86 simulations performed here required a total time of 31 hr. 35 min.

With Surrounding Direct Fire, the nonlinearity in (5.1) makes an army's attack stronger when the army is spread out in space. But direct fire is most effective when an army is concentrated near its enemy. There is a tradeoff between the advantages of concentration and dispersion, and the optimal placement of an army is somewhere in between.

6 Conclusions

In this paper we have extended Lanchester's (ordinary differential) Equations to partial differential equations with variables of time and two spatial dimensions. Four attrition rules were studied, each based on Lanchester's direct fire and indirect fire rules. The behaviors of armies under these rules were tested numerically. It would be worthwhile to rigorously establish the existence of solutions to the PDE analytically.

Local Direct Fire is similar to Lanchester's direct fire rule, except that it must be tweaked a little in order to avoid negative army densities. The army that concentrates its forces more tightly wins. One drawback of this method as formulated here is that there is no limit to how large the density of an army can get. It would be worthwhile, and add realism, to introduce a mechanism for penalizing an army for overconcentration. Assuming some velocity limit on the armies' movement about the plane, it would also be interesting to determine the optimal amount of time an army should spend concentrating itself before engaging an enemy with similar goals.

Nonlocal Area Fire is like Lanchester's area fire except that attrition is inflicted from a distance. Attrition is proportional to the product of the victim army and a convolution of the attacking army with a bivariate Gaussian. Each army's attacking Gaussian profile has a standard deviation and a magnitude. If the standard deviations for two armies are the same, the armies' populations follow Lanchester's linear law. If they are different, then the outcome of battle depends on the relative positions of the armies and their movement. It would be interesting to find optimal strategies of movement for armies with differing standard deviations.

Nonlocal Direct Fire is a version of Lanchester's Direct Fire in which attrition is inflicted from a distance. Each combat unit picks from among those targets attainable by it and fires at them in proportion with their densities. Unfortunately, the multiple convolutions involved in the attrition terms of the differential equations necessitated a symmetry assumption on the armies, effectively reducing the number of spatial dimensions to one. As in the Local Direct Fire model, concentrating an army results in a more effective attack. The same open questions that have been posed for Local Direct Fire apply to Nonlocal Direct Fire.

The most complicated, and perhaps most realistic, method of attack is Surrounding Direct Fire. As in Nonlocal Direct Fire, each combat unit picks from among those targets attainable by it and fires at them in proportion with their densities. But here the attrition suffered by the enemy is not simply the sum of enemy fire, but is computed by a formula that rewards the attacker for being spread out in space. However, if an army is too spread out, its attack is ineffective. A numerical experiment showed that there is a tradeoff between underconcentration and overconcentration, with an optimum level of concrentration somewhere in between. Again, imposing some limit on the velocity vector field for an army's motion, it would be interesting to seek optimal strategies for movement and reacting to a similarly equipped army.

In summation, Lanchester's equations can be extended to the space-time domain in several different ways, resulting in partial differential equations. The equations are relatively simple when compared to other simulation methods, and are amenable to analytical treatment and numerical experiment. Some of these models are complex enough that it is an interesting problem to seek optimal strategies for the combatants.

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