2006

On Doubly Periodic Solutions of Quasilinear Hyperbolic Equations of the Fourth Order

T. Kiguradze
Florida Institute of Technology

T. Smith
Florida Institute of Technology, smitht1@erau.edu

This paper is part of the conference proceedings edited by Ravi P. Agarwal and Kanishka Perera, and published by Hindawi Publishing Corporation, in 2006. ISBN: 977-5945-380.

ON DOUBLY PERIODIC SOLUTIONS OF QUASILINEAR HYPERBOLIC EQUATIONS OF THE FOURTH ORDER

T. KIGURADZE AND T. SMITH

The problem on doubly periodic solutions is considered for a class of quasilinear hyperbolic equations. Effective sufficient conditions of solvability and unique solvability of this problem are established.

The problem on periodic solutions for second-order partial differential equations of hyperbolic type has been studied rather intensively by various authors [1–9, 11–14]. Analogous problem for higher-order hyperbolic equations is little investigated. In the present paper for the quasilinear hyperbolic equations

$$u^{(2,2)} = f_0(x,y,u) + f_1(y,u)u^{(2,0)} + f_2(x,u)u^{(0,2)} + f\left(x,y,u,u^{(1,0)},u^{(0,1)},u^{(1,1)}\right), \tag{1}$$

$$\begin{aligned} u^{(2,2)} = {} & f_0(x,y,u) + \left(f_1(x,y,u)u^{(1,0)}\right)^{(1,0)} + \left(f_2(x,y,u)u^{(0,1)}\right)^{(0,1)} \\ & + f\left(x,y,u,u^{(1,0)},u^{(0,1)},u^{(1,1)}\right) \end{aligned} \tag{2}$$

we consider the problem on doubly periodic solutions

$$u(x+\omega_1,y) = u(x,y), \quad u(x,y+\omega_2) = u(x,y) \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{3}$$

Here ω_1 and ω_2 are prescribed positive numbers,

$$u^{(j,k)}(x,y) = \frac{\partial^{j+k}u(x,y)}{\partial x^j \partial y^k}, \tag{4}$$

$f_0(x,y,z)$, $f_1(y,z)$, $f_2(x,z)$, $f_1(x,y,z)$, $f_2(x,y,z)$, and $f(x,y,z,z_1,z_2,z_3)$ are continuous functions, ω_1-periodic in x, and ω_2-periodic in y.

Hindawi Publishing Corporation
Proceedings of the Conference on Differential & Difference Equations and Applications, pp. 541–553

This problem was studied thoroughly for the linear equation

$$u^{(2,2)} = p_0(x,y)u + p_1(x,y)u^{(2,0)} + p_2(x,y)u^{(0,2)} + q(x,y) \tag{5}$$

in [10]. The goal of the present paper is on the basis of the methods developed in [10] to obtain effective sufficient conditions of solvability, unique solvability, and well-posedness of problems (1), (3) and (2), (3).

Throughout the paper, we will use the following notation:

$$\operatorname{sgn}(z) = \begin{cases} 1, & z > 1, \\ 0, & z = 0, \\ -1, & z < 0. \end{cases} \tag{6}$$

$C^{m,n}_{\omega_1\omega_2}(\mathbb{R}^2)$ is the space of continuous functions $z : \mathbb{R}^2 \to \mathbb{R}$ ω_1-periodic in the first and ω_2-periodic in the second arguments, having the continuous partial derivatives $u^{(j,k)}$ $j \in \{0,\dots,m\}$, $k \in \{0,\dots,n\}$, with the norm

$$\|z\|_{C^{m,n}_{\omega_1\omega_2}} = \sup\left\{ \sum_{j=0}^{m}\sum_{k=0}^{n} \left| z^{(j,k)}(x,y)\right| : (x,y) \in \mathbb{R}^2 \right\}. \tag{7}$$

$L^2_{\omega_1\omega_2}(\mathbb{R}^2)$ is the space of locally square-integrable functions $z : \mathbb{R}^2 \to \mathbb{R}$, ω_1-periodic in the first and ω_2-periodic in the second arguments, with the norm

$$\|z\|_{L^2_{\omega_1\omega_2}} = \left(\int_0^{\omega_1}\int_0^{\omega_2} |z(s,t)|^2 ds\,dt \right)^{1/2}. \tag{8}$$

$H^{m,n}_{\omega_1\omega_2}(\mathbb{R}^2)$ is the space of functions $z \in L^2_{\omega_1\omega_2}(\mathbb{R}^2)$, having the generalized partial derivatives $u^{(j,k)} \in L^2_{\omega_1\omega_2}(\mathbb{R}^2)$, $j \in \{0,\dots,m\}$, $k \in \{0,\dots,n\}$, with the norm

$$\|z\|_{H^{m,n}_{\omega_1\omega_2}} = \sum_{j=0}^{m}\sum_{k=0}^{n} \left\| u^{(j,k)} \right\|_{L^2_{\omega_1\omega_2}}. \tag{9}$$

By a solution of problem (1), (3) (problem (2), (3)), we understand a classical solution, that is, a function $u \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ satisfying (1) (equation (2)) everywhere in $\mathbb{R}^2$.

Theorem 1. *Let there exists a positive constant δ such that*

$$f_1(y,z) \geq \delta, \quad f_2(x,z) \geq \delta \quad \text{for } (x,y,z) \in \mathbb{R}^3. \tag{10}$$

Moreover let the functions f_1, f_2, f_0, and f satisfy the conditions

$$\big(f_1(y,z) - f_1(y,\bar{z})\big)\operatorname{sgn}(z-\bar{z})\operatorname{sgn}(z) \geq 0 \quad \textit{for } y \in \mathbb{R},\ z\bar{z} \geq 0, \tag{11}$$

$$\big(f_2(x,z) - f_2(x,\bar{z})\big)\operatorname{sgn}(z-\bar{z})\operatorname{sgn}(z) \geq 0 \quad \textit{for } x \in \mathbb{R},\ z\bar{z} \geq 0, \tag{12}$$

$$\begin{gathered} f_0(x,y,z)\operatorname{sgn}(z) < 0 \quad \textit{for } (x,y) \in \mathbb{R}^2,\ z \neq 0, \\ \lim_{z\to\infty} \operatorname{sgn}(z)\int_0^{\omega_1}\int_0^{\omega_2} f_0(x,y,z)\,dx\,dy = -\infty, \end{gathered} \tag{13}$$

$$\lim_{z\to\infty} \frac{f(x,y,z,z_1,z_2,z_3)}{f_0(x,y,z)} = 0 \quad \text{uniformly on } \mathbb{R}^2 \times \mathbb{R}^4. \tag{14}$$

Then problem (1), (3) is solvable.

Theorem 2. *Let f_1 and f_2 be continuously differentiable functions such that*

$$f_1(x,y,z) \geq \delta, \quad f_2(x,y,z) \geq \delta \quad \textit{for } (x,y,z) \in \mathbb{R}^3 \tag{15}$$

for some positive δ. Moreover, let the functions f_0 and f satisfy the conditions of Theorem 1. Then problem (2), (3) is solvable.

Remark 1. Note that conditions (10) and (15) are optimal in a sense that we cannot take $\delta = 0$. Indeed, consider the problems

$$u^{(2,2)} = -F(u) + \Big(F'(u)u^{(1,0)}\Big)^{(1,0)} + u^{(0,2)} + \pi\sin x, \tag{16}$$

$$u(x+2\pi,y) = u(x,y), \qquad u(x,y+2\pi) = u(x,y), \tag{17}$$

where $F(z) = z^3$, or $F(z) = \arctan(z)$. Problem (16), (17) satisfies all of the conditions of Theorem 2 except condition (15). Instead of (15), we have that $F'(z)$ is nonnegative and vanishes at 0, or at ∞ only.

Let us show that problem (16), (17) has no solution. Assume the contrary: let u be a solution of (16), (17), and set $v(x,y) = u^{(0,2)}(x,y) - F(u(x,y))$. Then for every $y \in \mathbb{R}$, the function $v(\cdot,y)$ is a solution to the periodic problem

$$v'' = v + \pi\sin x, \qquad v(x+2\pi) = v(x). \tag{18}$$

This problem has a unique solution $v(x) = -\pi/2\sin x$. Therefore, problem (16), (17) is equivalent to the problem

$$u^{(0,2)} = F_1(u) - \frac{\pi}{2}\sin x, \qquad u(x,y+2\pi) = u(x,y). \tag{19}$$

However, problem (19) has no more than one solution. Indeed, let u_1 and u_2 be arbitrary solutions to problem (19). Then one easily gets the identity

$$\int_0^{\omega_2}\Big(\Big(u_1^{(0,1)}(x,t) - u_2^{(0,1)}(x,t)\Big)^2 + \big(F(u_1(x,t)) - F(u_2(x,t))\big)\big(u_1(x,t) - u_2(x,t)\big)\Big)dt \equiv 0, \tag{20}$$

whence it follows that $u_1(x,y) \equiv u_2(x,y)$.

Due to uniqueness, a solution of problem (19) should be independent of y. So finally we arrive to the functional equation

$$F(u) = \frac{\pi}{2}\sin x, \tag{21}$$

whence we get

$$\begin{aligned} u(x,y) &= \sqrt[3]{\frac{\pi}{2}\sin x} \quad \text{for } F(z) = z^3, \\ u(x,y) &= \tan\left(\frac{\pi}{2}\sin x\right) \quad \text{for } F(z) = \arctan(z). \end{aligned} \tag{22}$$

In the first case u is not differentiable at πk, $k \in \mathbb{Z}$, while in the second case u itself is a discontinuous function, because it blows up at points $\pi/2 + \pi k$, $k \in \mathbb{Z}$.

Thus, it is clear that of problem (16), (17) has no solutions in the both cases.

Remark 2. The conditions of Theorem 1 (as well as Theorem 2) do not guarantee the uniqueness of a solution. Indeed, for the equation

$$u^{(2,2)} = -u^n + u^{(2,0)} + u^{(0,2)} - \left(\prod_{k=1}^{n}(u-k) - u^n\right), \tag{23}$$

all of the conditions of Theorem 1 (and Theorem 2) are fulfilled. Nevertheless, it has at least n solutions $u_k(x,y) \equiv k$ $(k = 1,2,\dots,n)$ satisfying conditions (3).

We will give a uniqueness theorem for the equations

$$u^{(2,2)} = f_0(x,y,u) + \left(f_1(x,y)u^{(1,0)}\right)^{(1,0)} + \left(f_2(x,y)u^{(0,1)}\right)^{(0,1)}, \tag{24}$$

$$\begin{aligned} u^{(2,2)} = {} & f_0(x,y,u) + \left(f_1(x,y)u^{(1,0)}\right)^{(1,0)} + \left(f_2(x,y)u^{(0,1)}\right)^{(0,1)} \\ & + \varepsilon f\left(x,y,u,u^{(1,0)},u^{(0,1)},u^{(1,1)},u^{(2,0)},u^{(0,2)},u^{(2,1)},u^{(1,2)}\right). \end{aligned} \tag{25}$$

Theorem 3. *Let there exists $\delta > 0$ such that*

$$f_1(x,y) \geq \delta, \quad f_2(x,y) \geq \delta \quad \textit{for } (x,y) \in \mathbb{R}^2, \tag{26}$$

$$\left(f_0(x,y,z) - f_0(x,y,\overline{z})\right)\operatorname{sgn}(z-\overline{z}) \leq -\delta|z-\overline{z}| \quad \textit{for } (x,y) \in \mathbb{R}^2,\ z,\overline{z} \in \mathbb{R}. \tag{27}$$

Then problem (24), (3) is uniquely solvable. Moreover, for every $f(x,y,z_1,z_2,z_3,z_4,z_5,z_6,z_7,z_8)$ that is Lipschitz continuous with respect to the last eight phase variables, there exists a positive ε_0 such that problem (25), (3) is uniquely solvable for every $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$.

To prove Theorems 1–3, we will need the following lemmas.

Lemma 1. *Let p_0, p_1, p_2, and $q \in C_{\omega_1\omega_2}(\mathbb{R}^2)$, and let there exist a positive constant δ and a nondecreasing continuous function $\eta : [0,+\infty) \to [0,+\infty)$, $\eta(0) = 0$ such that*

$$p_1(x,y) \geq \delta, \qquad p_2(x,y) \geq \delta, \tag{28}$$

$$\begin{aligned} &|\,p_1(x_1,y_1) - p_1(x_2,y_2)\,| + |\,p_2(x_1,y_1) - p_2(x_2,y_2)\,| \\ &\quad \leq \eta(\,|\,x_1 - x_2\,| + |\,y_1 - y_2\,|\,) \quad \text{for } (x_i,y_i) \in \mathbb{R}^2 \ (i = 1,2). \end{aligned} \tag{29}$$

Then an arbitrary solution u of problem (5), (3) admits the estimate

$$\begin{aligned} &\int_0^{\omega_1}\int_0^{\omega_2}\Big(\left|u^{(2,0)}(x,y)\right|^2 + \left|u^{(0,2)}(x,y)\right|^2 + \left|u^{(2,1)}(x,y)\right|^2 + \left|u^{(1,2)}(x,y)\right|^2\Big)dx\,dy \\ &\quad \leq M\int_0^{\omega_1}\int_0^{\omega_2}\Big(|\,u(x,y)\,|^2 + \left|u^{(1,0)}(x,y)\right|^2 + \left|u^{(0,1)}(x,y)\right|^2 + q^2(x,y)\Big)dx\,dy, \end{aligned} \tag{30}$$

where the constant $M > 0$ depends on δ, $\|p_0\|_{C_{\omega_1\omega_2}}$, and the function η.

Proof. Let u be a an arbitrary solution of problem (5), (3). For any $h > 0$, set

$$\begin{aligned} &p_{ih}(x,y) = \frac{1}{h^2}\int_x^{x+h}\int_y^{y+h} p_i(s,t)\,ds\,dt \quad (i = 1,2), \\ &Q_h[u](x,y) = (p_1(x,y) - p_{1h}(x,y))u^{(2,0)}(x,y) + (p_2(x,y) - p_{2h}(x,y))u^{(0,2)}(x,y). \end{aligned} \tag{31}$$

Then u satisfies the equation

$$u^{(2,2)} = p_0(x,y)u + p_{1h}(x,y)u^{(2,0)} + p_{2h}(x,y)u^{(0,2)} + Q_h[u](x,y) + q(x,y). \tag{32}$$

Multiplying successively (32) by $u(x,y)$, $u^{(2,0)}$, and $u^{(0,2)}$, integrating over the rectangle $[0,\omega_1] \times [0,\omega_2]$, and using integration by parts, we observe that

$$\begin{aligned} &\int_0^{\omega_1}\int_0^{\omega_2}\Big(p_{1h}(x,y)\left|u^{(1,0)}(x,y)\right|^2 + p_{2h}(x,y)\left|u^{(0,1)}(x,y)\right|^2 + \left|u^{(1,1)}(x,y)\right|^2\Big)dx\,dy \\ &\quad = \int_0^{\omega_1}\int_0^{\omega_2}\Big(Q_h[u](x,y) - p_{1h}^{(1,0)}(x,y)u^{(1,0)}(x,y) - p_{2h}^{(0,1)}(x,y)u^{(0,1)}(x,y) \\ &\qquad + p_0(x,y)u(x,y) + q(x,y)\Big)u(x,y)\,dx\,dy, \end{aligned} \tag{33}$$

$$\begin{aligned}
&\int_0^{\omega_1}\int_0^{\omega_2}\Big(p_{1h}(x,y)\left|u^{(2,0)}(x,y)\right|^2+p_{2h}(x,y)\left|u^{(1,1)}(x,y)\right|^2+\left|u^{(2,1)}(x,y)\right|^2\Big)dx\,dy\\
&\quad=\int_0^{\omega_1}\int_0^{\omega_2}\Big(p_{2h}^{(0,1)}(x,y)u^{(2,0)}(x,y)u^{(0,1)}(x,y)-p_{2h}^{(1,0)}(x,y)u^{(1,1)}(x,y)u^{(0,1)}(x,y)\Big)dx\,dy\\
&\qquad-\int_0^{\omega_1}\int_0^{\omega_2}\big(Q_h[u](x,y)+p_0(x,y)u(x,y)+q(x,y)\big)u^{(2,0)}(x,y)dx\,dy,
\end{aligned}\tag{34}$$

$$\begin{aligned}
&\int_0^{\omega_1}\int_0^{\omega_2}\Big(p_{1h}(x,y)\left|u^{(1,1)}(x,y)\right|^2+p_{2h}(x,y)\left|u^{(0,2)}(x,y)\right|^2+\left|u^{(1,2)}(x,y)\right|^2\Big)dx\,dy\\
&\quad=\int_0^{\omega_1}\int_0^{\omega_2}\Big(p_{1h}^{(1,0)}(x,y)u^{(0,2)}(x,y)u^{(1,0)}(x,y)-p_{1h}^{(0,1)}(x,y)u^{(1,1)}(x,y)u^{(1,0)}(x,y)\Big)dx\,dy\\
&\qquad-\int_0^{\omega_1}\int_0^{\omega_2}\big(Q_h[u](x,y)+p_0(x,y)u(x,y)+q(x,y)\big)u^{(0,2)}(x,y)dx\,dy.
\end{aligned}\tag{35}$$

However,

$$\begin{aligned}
&\int_0^{\omega_1}\int_0^{\omega_2}\left|Q_h[u](x,y)\right|\Big(|u(x,y)|+\left|u^{(2,0)}(x,y)\right|+\left|u^{(0,2)}(x,y)\right|\Big)dx\,dy\\
&\quad\le 2\eta(h)\Big(\|u\|^2_{L^2_{\omega_1\omega_2}}+\left\|u^{(2,0)}\right\|^2_{L^2_{\omega_1\omega_2}}+\left\|u^{(0,2)}\right\|^2_{L^2_{\omega_1\omega_2}}\Big),
\end{aligned}\tag{36}$$

$$\begin{aligned}
&\int_0^{\omega_1}\int_0^{\omega_2}\big(|p_0(x,y)|\,|u(x,y)|+|q(x,y)|\big)\Big(|u(x,y)|+\left|u^{(2,0)}(x,y)\right|+\left|u^{(0,2)}(x,y)\right|\Big)dx\,dy\\
&\quad\le\Big(\frac{2}{\varepsilon}\|p_0\|_{C_{\omega_1\omega_2}}+2\varepsilon\Big)\|u\|^2_{L^2_{\omega_1\omega_2}}+\frac{2}{\varepsilon}\|q\|^2_{L^2_{\omega_1\omega_2}}+2\varepsilon\Big(\left\|u^{(2,0)}\right\|^2_{L^2_{\omega_1\omega_2}}+\left\|u^{(0,2)}\right\|^2_{L^2_{\omega_1\omega_2}}\Big),
\end{aligned}\tag{37}$$

$$\begin{aligned}
&\int_0^{\omega_1}\int_0^{\omega_2}\Big(\left|p_{2h}^{(0,1)}(x,y)\right|\left|u^{(2,0)}(x,y)\right|\left|u^{(0,1)}(x,y)\right|\\
&\qquad+\left|p_{2h}^{(1,0)}(x,y)\right|\left|u^{(1,1)}(x,y)\right|\left|u^{(0,1)}(x,y)\right|\Big)dx\,dy\\
&\quad\le\frac{2\eta(h)}{h}\varepsilon\Big(\left\|u^{(2,0)}\right\|^2_{L^2_{\omega_1\omega_2}}+\left\|u^{(1,1)}\right\|^2_{L^2_{\omega_1\omega_2}}\Big)+\frac{2\eta(h)}{h\varepsilon}\left\|u^{(0,1)}\right\|^2_{L^2_{\omega_1\omega_2}},
\end{aligned}\tag{38}$$

$$\begin{aligned}
&\int_0^{\omega_1}\int_0^{\omega_2}\Big(\left|p_{1h}^{(1,0)}(x,y)\right|\left|u^{(0,2)}(x,y)\right|\left|u^{(1,0)}(x,y)\right|\\
&\qquad+\left|p_{1h}^{(0,1)}(x,y)\right|\left|u^{(1,1)}(x,y)\right|\left|u^{(1,0)}(x,y)\right|\Big)dx\,dy\\
&\quad\le\frac{2\eta(h)}{h}\varepsilon\Big(\left\|u^{(0,2)}\right\|^2_{L^2_{\omega_1\omega_2}}+\left\|u^{(1,1)}\right\|^2_{L^2_{\omega_1\omega_2}}\Big)+\frac{2\eta(h)}{h\varepsilon}\left\|u^{(1,0)}\right\|^2_{L^2_{\omega_1\omega_2}}.
\end{aligned}\tag{39}$$

Now taking $h>0$ and $\varepsilon>0$ sufficiently small from (33)–(39), we immediately get estimate (30). □

The following lemma immediately follows from [10, Lemma 2.7].

LEMMA 2. *Let p_0, p_1, p_2, and $q \in C_{\omega_1\omega_2}(\mathbb{R}^2)$, and let p_1 and p_2 satisfy conditions (28). Then an arbitrary solution u of problem (5), (3) admits the estimate*

$$\|u\|_{C^{2,2}_{\omega_1\omega_2}} \le r\left(\int_0^{\omega_1}\int_0^{\omega_2}\left(|u(x,y)| + \left|u^{(2,0)}(x,y)\right| + \left|u^{(0,2)}(x,y)\right|\right)dx\,dy + \|q\|_{C_{\omega_1\omega_2}}\right), \tag{40}$$

where r is a positive constant depending on δ, $\|p_0\|_{C_{\omega_1\omega_2}}$, $\|p_1\|_{C_{\omega_1\omega_2}}$, and $\|p_2\|_{C_{\omega_1\omega_2}}$ only.

LEMMA 3. *Let $p_1, p_2 \in C_{\omega_1\omega_2}(\mathbb{R}^2)$ satisfy the conditions of Lemma 1. Then there exist $\lambda > 0$ and $M_\lambda > 0$ depending on δ, $\|p_1\|_{C_{\omega_1\omega_2}}$, $\|p_2\|_{C_{\omega_1\omega_2}}$, and the function η such that for every $q \in C_{\omega_1\omega_2}(\mathbb{R}^2)$, the equation*

$$u^{(2,2)} = -\lambda u + p_1(x,y)u^{(2,0)} + p_2(x,y)u^{(0,2)} + q(x,y) \tag{41}$$

has a unique solution u satisfying conditions (3), and

$$\|u\|_{C^{2,2}_{\omega_1\omega_2}} \le M_\lambda\|q\|_{C_{\omega_1\omega_2}}. \tag{42}$$

Proof. This lemma easily follows from Lemmas 1 and 2. Indeed, let u be an arbitrary solution of problems (41), (3). Multiplying successively (41) by $u(x,y)$, $u^{(2,0)}$, and $u^{(0,2)}$, integrating over the rectangle $[0,\omega_1]\times[0,\omega_2]$, and using integration by parts, we get

$$\begin{aligned}
&\lambda\left(\|u\|^2_{L^2_{\omega_1\omega_2}} + \|u^{(1,0)}\|^2_{L^2_{\omega_1\omega_2}} + \|u^{(0,1)}\|^2_{L^2_{\omega_1\omega_2}}\right)\\
&\quad\le \left(1 + \|p_1\|^2_{C_{\omega_1\omega_2}} + \|p_2\|^2_{C_{\omega_1\omega_2}}\right)\left(\|u\|^2_{L^2_{\omega_1\omega_2}} + \|u^{(2,0)}\|^2_{L^2_{\omega_1\omega_2}} + \|u^{(0,2)}\|^2_{L^2_{\omega_1\omega_2}}\right) + \|q\|^2_{L^2_{\omega_1\omega_2}}.
\end{aligned} \tag{43}$$

Validity of Lemma 3 immediately follows from estimates (30), (40), and (43). □

Consider the linear equation

$$u^{(2,2)} = p_0(x,y)u + \left(p_1(x,y)u^{(1,0)}\right)^{(1,0)} + \left(p_2(x,y)u^{(0,1)}\right)^{(0,1)} + q(x,y). \tag{44}$$

If p_1 and p_2 satisfy (28), then by $g_1(\cdot,\cdot,x):\mathbb{R}^2\to\mathbb{R}$ and $g_2(\cdot,\cdot,y):\mathbb{R}^2\to\mathbb{R}$, respectively, denote Green's functions of the problems

$$\begin{aligned}
\frac{d^2z}{dy^2} &= p_1(x,y)z, \qquad z(y+\omega_2) = z(y),\\
\frac{d^2z}{dx^2} &= p_2(x,y)z, \qquad z(x+\omega_1) = z(x),
\end{aligned} \tag{45}$$

(see [10, Lemmas 2.1 and 2.2]).

Lemma 4. *Let u be a solution of problem (44), (3). Then the following representation is valid*

$$
\begin{aligned}
u^{(2,0)}(x,y) &= p_2(x,y)u \\
&\quad + \int_y^{y+\omega_2} g_1(y,t,x)\Big(\big(p_0(x,t)+p_1(x,t)p_2(x,t)\big)u(x,t) \\
&\qquad\qquad + p_1^{(1,0)}(x,t)u^{(1,0)}(x,t)+q(x,t)\Big)dt, \\
u^{(0,2)}(x,y) &= p_1(x,y)u \\
&\quad + \int_x^{x+\omega_1} g_2(x,s,y)\Big(\big(p_0(s,y)+p_1(s,y)p_2(s,y)\big)u(s,y) \\
&\qquad\qquad + p_2^{(0,1)}(s,y)u^{(0,1)}(s,y)+q(s,y)\Big)ds, \\
u(x,y) &= \int_y^{y+\omega_2}\int_x^{x+\omega_1} g_1(y,t,x)g_2(x,s,t)\Big(\big(p_0(s,t)+p_1(s,t)p_2(s,t)\big)u(s,t) \\
&\qquad\qquad + p_2^{(0,1)}(s,t)u^{(0,1)}(s,t)+q(s,t)\Big)ds\,dt.
\end{aligned}
\tag{46}
$$

We omit the proof of Lemma 4, since it is similar to the proof of [10, Lemma 2.7]. Let

$$
\varphi_\rho(z) = \begin{cases} 1 & \text{for } |z| \le \rho, \\ \rho+1-|z| & \text{for } |z| \in [\rho,\rho+1], \\ 0 & \text{for } |z| \ge \rho+2, \end{cases} \qquad \chi_\rho(z) = \int_0^z \varphi_\rho(s)ds, \tag{47}
$$

and let $\Phi_\rho : C^1_{\omega_1\omega_2} \to \mathbb{R}$ be a continuous nonlinear functional defined by the equality

$$
\Phi_\rho(u) = \varphi_\rho\Big(\|u\|_{C^1_{\omega_1\omega_2}}\Big). \tag{48}
$$

Consider the equation

$$
\begin{aligned}
u^{(2,2)} &= f_0(x,y,\chi_\rho(u)) + f_1(y,\Phi_\rho(u)u)u^{(2,0)} + f_2(x,\Phi_\rho(u))u^{(0,2)} \\
&\quad + \Phi_\rho(u) f\Big(x,y,u,u^{(1,0)},u^{(0,1)},u^{(1,1)}\Big) - \lambda u + \lambda\chi_\rho(u).
\end{aligned}
\tag{49}
$$

Lemma 5. *Let $\lambda > 0$ and $\rho > 0$. Then every solution u of problem (49), (3) admits the estimates*

$$
\begin{aligned}
\int_0^{\omega_1}\int_0^{\omega_2} \Big(&|f_0(x,y,\chi_\rho(u(x,y)))|\,|u(x,y)| + \Big|u^{(1,0)}(x,y)\Big|^2 \\
&+ \Big|u^{(0,1)}(x,y)\Big|^2 + \Big|u^{(1,1)}\Big|^2\Big)dx\,dy \le r_0,
\end{aligned}
\tag{50}
$$

where r_0 is a positive constant independent of ρ, λ, and u.

Proof. Let u be a solution of problems (49), (3). Multiplying (49) by $u(x,y)$, integrating over the rectangle $[0,\omega_1]\times[0,\omega_2]$, and using integration by parts, we get

$$\begin{aligned}\int_0^{\omega_1}\int_0^{\omega_2}\Big(&\big(-f_0(x,y,\chi_\rho(u(x,y)))+\lambda u-\lambda\chi_\rho(u)\big)u(x,y)-f_1(y,u(x,y))u^{(2,0)}(x,y)u(x,y)\\ &-f_2(x,u(x,y))u^{(0,2)}(x,y)u(x,y)+\left|u^{(1,1)}(x,y)\right|^2\Big)dx\,dy\\ =\int_0^{\omega_1}\int_0^{\omega_2}&\Phi_\rho(u)f\Big(x,y,u(x,y),u^{(1,0)}(x,y),u^{(0,1)}(x,y),u^{(1,1)}(x,y)\Big)u(x,y)dx\,dy.\end{aligned} \tag{51}$$

By conditions (13) and (14), we have

$$\begin{aligned}&\big(-f_0(x,y,\chi_\rho(u(x,y)))+\lambda(u(x,y)-\chi_\rho(u(x,y)))\big)u(x,y)\\ &\quad\geq\left|f_0(x,y,\chi_\rho(u(x,y)))u(x,y)\right|,\end{aligned} \tag{52}$$

$$\begin{aligned}&\Phi_\rho(u)\left|f\Big(x,y,u(x,y),u^{(1,0)}(x,y),u^{(0,1)}(x,y),u^{(1,1)}(x,y)\Big)\right|\left|u(x,y)\right|\\ &\quad\leq r_1+\frac{1}{2}\left|f_0(x,y,\chi_\rho(u(x,y)))\right|\left|u(x,y)\right|,\end{aligned} \tag{53}$$

where r_1 is a positive constant independent of ρ, λ, and u.

For $h>0$, set

$$f_{1h}(y,z)=\frac{1}{h}\int_z^{z+h}f_i(y,\xi)d\xi. \tag{54}$$

Then by condition (11), we have

$$\begin{aligned}-\int_0^{\omega_1}\int_0^{\omega_2}&f_{1h}(y,\Phi_\rho(u)u(x,y))u(x,y)u^{(2,0)}(x,y)dx\,dy\\ =\int_0^{\omega_1}\int_0^{\omega_2}&f_{1h}(y,\Phi_\rho(u)u(x,y))\left|u^{(1,0)}(x,y)\right|^2dx\,dy\\ &+\frac{\Phi_\rho(u)}{h}\int_0^{\omega_1}\int_0^{\omega_2}\Big(f_1(y,\Phi_\rho(u)(u(x,y)+h)\\ &\qquad-f_1(y,\Phi_\rho(u)u(x,y))\Big)u(x,y)\left|u^{(1,0)}(x,y)\right|^2dx\,dy\\ \geq\int_0^{\omega_1}\int_0^{\omega_2}&f_{1h}(y,\Phi_\rho(u)u(x,y))\left|u^{(1,0)}(x,y)\right|^2dx\,dy\\ &-\Phi_\rho(u)\iint_{D_h}\left|f_1(y,\Phi_\rho(u)(u(x,y)+h)-f_1(y,\Phi_\rho(u)u(x,y)\right|\left|u^{(1,0)}(x,y)\right|^2dx\,dy,\end{aligned} \tag{55}$$

where $D_h = \{(x,y) \in [0,\omega_1] \times [0,\omega_2] : |u(x,y)| \le h\}$. Hence we immediately get that

$$-\int_0^{\omega_1}\int_0^{\omega_2} f_1(y,\Phi_\rho(u)u(x,y))u(x,y)u^{(2,0)}(x,y)dx\,dy \\ \ge \int_0^{\omega_1}\int_0^{\omega_2} f_1(y,\Phi_\rho(u)u(x,y))\left|u^{(1,0)}(x,y)\right|^2 dx\,dy. \tag{56}$$

In the same way, we show that

$$-\int_0^{\omega_1}\int_0^{\omega_2} f_2(x,\Phi_\rho(u)u(x,y))u(x,y)u^{(0,2)}(x,y)dx\,dy \\ \ge \int_0^{\omega_1}\int_0^{\omega_2} f_2(y,\Phi_\rho(u)u(x,y))\left|u^{(0,1)}(x,y)\right|^2 dx\,dy. \tag{57}$$

Taking into account (52)–(57), from (51), we immediately get (50) with $r_0 = (2+\delta^{-1})r_1$. □

Proof of Theorem 1. Let $v \in C^{1,1}_{\omega_1\omega_2}(\mathbb{R}^2)$ be an arbitrary function. Set

$$p_1[v](x,y) = f_1(y,\Phi_\rho(v)v(x,y)), \qquad p_2[v](x,y) = f_2(x,\Phi_\rho(v)v(x,y)), \\ q[v](x,y) = f_0(X,y,\chi_\rho(v(x,y))) \\ +\Phi_\rho(v)f\left(x,y,v(x,y),v^{(1,0)}(x,y),v^{(0,1)}(x,y),v^{(1,1)}(x,y)\right). \tag{58}$$

Consider the equation

$$u^{(2,2)} = -\lambda u + p_1[v](x,y)u^{(2,0)} + p_2[v](x,y)u^{(0,2)} + \lambda\chi_\rho(v(x,y)) + q[v](x,y). \tag{59}$$

Note that due to definitions of p_1 and p_2 for every $\rho > 0$, there exists a continuous function $\eta_\rho : [0,+\infty) \to [0,+\infty)$, $\eta_\rho(0) = 0$ such that

$$|p_1[v](x_1,y_1) - p_2[v](x_2,y_2)| + |p_2[v](x_1,y_1) - p_2[v](x_2,y_2)| \\ \le \eta_\rho(|x_1-x_2| + |y_1-y_2|). \tag{60}$$

By Lemma 3, there exist $\lambda > 0$ and $M_\lambda > 0$ depending on ρ, δ, and the function η_ρ only, such that for every $v \in C^{1,1}_{\omega_1\omega_2}(\mathbb{R}^2)$, problem (59), (3) has a unique solution $u[v]$ admitting the estimate

$$||u[v]||_{C^{2,2}_{\omega_1\omega_2}} \le M_\lambda(||q[v]||_{C_{\omega_1\omega_2}} + \lambda\rho). \tag{61}$$

It is easy to see that the operator $\mathcal{A} : v \to u[v]$ is a continuous operator from $C^{1,1}_{\omega_1\omega_2}(\mathbb{R}^2)$ into $C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$, and therefore it is a completely continuous operator from $C^{1,1}_{\omega_1\omega_2}(\mathbb{R}^2)$ into $C^{1,1}_{\omega_1\omega_2}(\mathbb{R}^2)$. Moreover,

$$||\mathcal{A}(v)||_{C^{2,2}_{\omega_1\omega_2}} \le M_\lambda(||q[v]||_{C_{\omega_1\omega_2}} + \lambda\rho) \le M_\lambda c_\rho, \tag{62}$$

where c_ρ is a positive constant independent of v.

By Schauder's fixed point theorem, the operator $\mathcal{A}$ has a fixed point $u \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$, which is a solution of the functional differential equation (49).

By Lemma 5, u admits estimate (50). Conditions (13) and (50) imply the estimate

$$\|u\|_{H^{1,1}_{\omega_1\omega_2}} \leq r_1, \tag{63}$$

where r_1 is a positive constant independent of ρ, λ, and u. On the other hand, one can easily establish the inequalities

$$\|u\|_{C_{\omega_1\omega_2}} \leq \Omega\|u\|_{H^{1,1}_{\omega_1\omega_2}}, \tag{64}$$

$$|u(x_1,y_1) - u(x_2,y_2)| \leq \Omega\|u\|_{H^{1,1}_{\omega_1\omega_2}}\left(\sqrt{|x_1-x_2|} + \sqrt{|y_1-y_2|}\right), \tag{65}$$

where

$$\Omega = \frac{1}{\sqrt{\omega_1}} + \frac{1}{\sqrt{\omega_2}} + \frac{1}{\sqrt{\omega_1\omega_2}} + \sqrt{\omega_1} + \sqrt{\omega_2}. \tag{66}$$

Choosing $\rho > \Omega r_1$, we observe that u is a solution of the equation

$$\begin{aligned} u^{(2,2)} = {} & f_0(x,y,u) + f_1(y,\Phi_\rho(u)u)u^{(2,0)} + f_2(x,\Phi_\rho(u))u^{(0,2)} \\ & + \Phi_\rho(u)f\left(x,y,u,u^{(1,0)},u^{(0,1)},u^{(1,1)}\right). \end{aligned} \tag{67}$$

Due to (63) and (65), there exists a nondecreasing continuous function $\eta : [0,+\infty) \to [0,+\infty)$, $\eta(0) = 0$ independent of ρ such that

$$\begin{aligned} & |f_1(y_1,\Phi_\rho(u)u(x_1,y_1)) - f_1(y_2,\Phi_\rho(u)u(x_2,y_2))| \\ & \quad + |f_2(x_1,\Phi_\rho(u)u(x_1,y_1)) - f_2(x_2,\Phi_\rho(u)u(x_2,y_2))| \leq \eta(|x_1-x_2| + |y_1-y_2|). \end{aligned} \tag{68}$$

By Lemma 1 and inequality (68), there exists a positive constant M independent of ρ such that u admits the estimate (30). Choosing $\rho > \Omega(r_1 + M)$, we get that an arbitrary solution of problems (67), (3) satisfies the inequality

$$\|u\|_{C^1_{\Omega_1\omega_2}} < \rho. \tag{69}$$

Consequently u is a solution of problem (1), (3) too. □

We omit the proof of Theorem 2, since it can be proved in much the same way. The only difference is that instead of Lemmas 1–3 one should use Lemma 4 to get necessary a priori estimates.

Proof of Theorem 3. Let $q \in C_{\omega_1\omega_2}(\mathbb{R}^2)$. Consider the equation

$$u^{(2,2)} = f_0(x,y,u) + \left(f_1(x,y)u^{(1,0)}\right)^{(1,0)} + \left(f_2(x,y)u^{(0,1)}\right)^{(0,1)} + q(x,y). \tag{70}$$

By Theorem 2, problems (70), (3) are solvable. Let u_1 and u_2 be two arbitrary solutions of problems (70), (3), and let $v(x,y) = u_1(x,y) - u_2(x,y)$. Then applying (27), we easily get the inequality

$$\int_0^{\omega_1}\int_0^{\omega_2}\left(\delta v^2(x,y)+f_1(x,y)\left|v^{(1,0)}(x,y)\right|^2+f_2(x,y)\left|v^{(0,1)}(x,y)\right|^2\right)dx\,dy\le 0. \tag{71}$$

Hence it follows that $u_1(x,y)\equiv u_2(x,y)$.

Thus for every $q\in C_{\omega_1\omega_2}(\mathbb{R}^2)$, problem (70), (3) has a unique solution $u[q]$. Applying Lemmas 1 and 2, one can easily show that the operator $\mathcal{A}:q\to u[q]$ is a continuous operator from $C_{\omega_1\omega_2}(\mathbb{R}^2)$ into $C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ and that

$$||\mathcal{A}(q_1)-\mathcal{A}(q_2)||_{C^{2,2}_{\omega_1\omega_2}}\le a||q_1-q_2||_{C_{\omega_1\omega_2}}, \tag{72}$$

where a is a positive constant independent of q_1 and q_2. Therefore problem (25), (3) is equivalent to the operator equation

$$u(x,y)=\mathcal{A}\left(\varepsilon f\left(x,y,u,u^{(1,0)},u^{(0,1)},u^{(1,1)},u^{(2,0)},u^{(0,2)},u^{(2,1)},u^{(1,2)}\right)\right)(x,y)=\mathcal{B}_\varepsilon(u)(x,y). \tag{73}$$

Due to Lipschitz continuity of the function f, there exists a positive constant b such that

$$|f(x,y,z_1,\dots,z_8)-f(x,y,\overline{z}_1,\dots,\overline{z}_8)|\le b\sum_{i=1}^{8}|z_i-\overline{z}_i|. \tag{74}$$

From (72) and (74), it is clear that for $\varepsilon\in(-1/ab,1/ab)$, the operator $\mathcal{B}_\varepsilon$ is a contractive operator from $C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ into $C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$. Hence (73), and consequently, problem (25), (3) is uniquely solvable. □

References

[1] A. K. Aziz and S. L. Brodsky, *Periodic solutions of a class of weakly nonlinear hyperbolic partial differential equations*, SIAM Journal on Mathematical Analysis **3** (1972), no. 2, 300–313.

[2] A. K. Aziz and M. G. Horak, *Periodic solutions of hyperbolic partial differential equations in the large*, SIAM Journal on Mathematical Analysis **3** (1972), no. 1, 176–182.

[3] L. Cesari, *Existence in the large of periodic solutions of hyperbolic partial differential equations*, Archive for Rational Mechanics and Analysis **20** (1965), no. 2, 170–190.

[4] ———, *Periodic solutions of nonlinear hyperbolic differential equations*, Colloques Internationaux du Centre National de la Recherche Scientifique **148** (1965), 425–437.

[5] ———, *Smoothness properties of periodic solutions in the large of nonlinear hyperbolic differential systems*, Funkcialaj Ekvacioj. Serio Internacia **9** (1966), 325–338.

[6] T. Kiguradze, *Some boundary value problems for systems of linear partial differential equations of hyperbolic type*, Memoirs on Differential equations and Mathematical Physics **1** (1994), 1–144.

[7] ———, *On periodic in the plane solutions of second order linear hyperbolic systems*, Archivum Mathematicum **33** (1997), no. 4, 253–272.

[8] ———, *Doubly periodic solutions of a class of nonlinear hyperbolic equations*, Differential Equations **34** (1998), no. 2, 242–249, translation from Differentsial'nye Uravneniya **34** (1998), no. 2, 238–245.

[9] ———, *On periodic in the plane solutions of nonlinear hyperbolic equations*, Nonlinear Analysis, Series A: Theory Methods **39** (2000), no. 2, 173–185.

[10] T. Kiguradze and V. Lakshmikantham, *On doubly periodic solutions of fourth-order linear hyperbolic equations*, Nonlinear Analysis, Series A: Theory Methods **49** (2002), no. 1, 87–112.
[11] B. P. Liu, *The integral operator method for finding almost periodic solutions of nonlinear wave equations*, Nonlinear Analysis. Theory, Methods & Applications **11** (1987), no. 5, 553–564.
[12] Yu. O. Mitropol's'kiĭ, G. P. Khoma, and P. V. Tsinaĭko, *A periodic problem for an inhomogeneous equation of string vibration*, Ukraïns'kiĭ Matematichniĭ Zhurnal **49** (1997), no. 4, 558–565 (Ukrainian).
[13] N. A. Perestyuk and A. B. Tkach, *Periodic solutions of a weakly nonlinear system of partial differential equations with impulse action*, Ukraïns'kiĭ Matematichniĭ Zhurnal **49** (1997), no. 4, 601–605 (Russian).
[14] B. I. Ptashnik, *Ill-Posed Boundary Value Problems for Partial Differential Equations*, Naukova Dumka, Kiev, 1984.

T. Kiguradze: Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA
E-mail address: tkigurad@fit.edu

T. Smith: Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA
E-mail address: smitht@fit.edu