Undergraduate's Covariational Reasoning Across Function Representations

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Covariational Reasoning is the mental actions, constructions, and processes used to coordinate two or more quantities and interpret the relation between them. While research has shown that covariational reasoning is critical in a variety of fields, there has been a lack of studies on three-dimensional covariational reasoning. This study utilizes the Action-Process-Object-Schema (APOS) Theory framework to analyze how a student applies covariational reasoning to a parametric representation to model a real-life three-dimensional scenario. Preliminary results suggest that students' focus on experiential time may inhibit their ability to reason about two or three quantities relating to each other irrespective to time.

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Introduction

Covariational reasoning is the mental actions, constructions, and processes used to coordinate two quantities and interpret the relation between them (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Carlson et al. (2002) showed students with strong covariational reasoning ability, but no calculus background, were able to complete the same limits and differentiation tasks that a group of second semester calculus students struggled with. While research has shown that covariational reasoning is critical in a variety of fields, there are still unknowns such as: (1) the mental processes students enact to understand covariational reasoning, (2) the foundations students need to develop their reasoning abilities, and (3) how students are able to apply covariational reasoning in different environments.

The goal of this study is to expand the existing literature by focusing on how students apply covariational reasoning in a different environment in comparison to previous studies. To accomplish this end, this study will focus on analyzing how students are able to interact and understand a three-dimensional model which utilizes a mixture of linear and nonlinear functions. The overarching question the study aims to answer is as follows:

> *How are students able to use covariational reasoning to create a parametric representation to model a real-life 3-dimensional problem?*

Literature Review

Over the past several decades, numerous articles and studies focusing on the role covariational reasoning plays in various contexts have been published, such as:

- The initial study from Carlson et al. (2002) which focused on coordinating quantity changes related to instantaneous rate of change;
- Using trigonometric functions to relate radian measures to arc lengths in a circle context (Moore & LaForest, 2014);
- Reasoning about two quantities through time as a third parameter (Paoletti & Moore, 2017); and
- Examining the role of reasoning about magnitudes when graphically representing covarying quantities (Moore, Stevens, Paoletti, Hobson, & Liang, 2019).

As our study focuses on a three-dimensional model, we considered two forms of reasoning associated with covariational reasoning: simultaneous-independent reasoning and change-dependent reasoning. *Simultaneous-independent reasoning* focuses on how two quantities vary with respect to a third quantity, which frequently is time (Stalvey & Vidakovic, 2015). This definition is particularly relevant to the discussion on parametric representations since the focus is describing how two or more functions vary relative to one another expressed through coordinates of the points. While these functions normally have the same input, the changes that occur in one function do not directly cause the changes in another function. *Change-dependent reasoning* focuses on how a quantity directly causes changes in a different quantity (Stalvey & Vidakovic, 2015), such as how the height of an object may change over time.

Theoretical Framework

This study will utilize the Action-Process-Object-Schema (APOS) Theory framework to analyze and interpret our results. APOS Theory describes the mental structures (Actions, Process, Objects, and Schemas) that individuals construct to learn a mathematical concept. Developing these structures are considered stages in the learning process (Arnon, et al., 2013). We briefly describe each construct below using definitions from Arnon et al. (2013).

When an individual first learns a new concept they start at the Action stage, which is described as when an individual can take a mathematical object and perform an explicit transformation based on external cues. These actions can be simple or complex, depending on the objects they are acting upon. After repeating an Action, individuals move away from relying on external cues and can control the procedure internally. At the Process stage, individuals can implicitly carry out the transformation and even deviate from the external cues they previously relied on. Students who can then act on this dynamic, internal procedure as a static object are said to be at the Object stage. These now-static objects can then be acted on by new external cues to continue developing the concept. Finally, a Schema is an ever-changing mental structure that an individual constructs and reconstructs. Schemas include Actions, Processes, Objects, and other Schemas about a single mathematical concept. Schema development occurs both through the stages an individual may take through a concept as well as through the connections between other mental structures related to the concept.

Methodology

To examine students' covariational reasoning while modeling a real-life problem, we developed a virtual model of a bird flying in a helix pattern around a tower. The student could change their view of the tower without interrupting the bird flight by rotating horizontally and vertically around the tower. The student could pause the bird's flight or leave it to loop. A flag was presented on the ground parallel to the tower to provide an additional landmark the student may use to reason through the variations in the bird's horizontal, vertical, and height displacement. After being showed how to change the views of the model, a student was prompted to answer questions along two separate goals: (task 1) graph an individual quantity with respect to time and (task 2) graph two or three quantities irrespective to time.

Prior to completing the task, students were asked to supply their personal definition for *function* and *derivative*. After completing the task and discussing their answers with the interviewer, students were again able to present their definitions for *function* and *derivative* to see if their definition had developed.

Volunteers to complete the interview were solicited from a Calculus 1 course at a large southeastern university during Spring 2021. One student volunteered to participate: pseudonym

Jane. Jane was a first-year university student, who (at the time of the study) was taking calculus for the third time. This was her second time taking it on the university level, prior to which they had taken a first semester calculus course through the International Baccalaureate program at their secondary institution.

Data Analysis

At the action level understanding of a *function*, in the first task we would expect a student to be reliant on selecting specific moments in time and their corresponding height to create their graph and recognize a linear relationship. Whereas students possessing a process level understanding would abstract this point process to create a smooth representation and a continuous line.

For the second task, students who understand covariational reasoning at the action level would start graphing individual points and connecting them to create their parametric representation. Students with a strong definition of derivative could recognize that as they plot more points, if they were to plot infinitely many points, then a smooth representation could emerge, which students could internalize to recognize how changes in one function coordinate with another function and thus attain a process level understanding of covariation reasoning.

After transcribing the interview, the authors analyzed Jane's responses to the task to identify evidence for simultaneous-independent and/or change-dependent reasoning in terms of APOS Theory. We present preliminary results from this analysis.

Results

Based upon the student's personal definition of *function* and their response to question 1, it was evident the student possessed an action level understanding of *function*. This caused them difficulties in coordinating each function to create a polar representation, which in conjunction with their definition of *derivative*, showed they also possessed an action level understanding of covariational reasoning. We present evidence for her level of conception through how she created a linear representation and a parametric representation of the bird's flight.

Creating a Linear Representation

A feature of the instrument that was not implemented was an explicit measurement tool, whether for measuring time or for determining numerical values for the bird's position. This meant if students depended on having specific inputs for determining a function representation or constructing a parametric representation, students would need to create their own measurement tool, which is what Jane did. As shown in the following image, Jane not only measured the period of the bird's flight around the tower, but also constructed ratios between the height of the bird to the flag to the tower. These ratios were indicated in her scratchwork and during the interview where she detailed using a piece of paper to measure the differences in height to create her ratio.

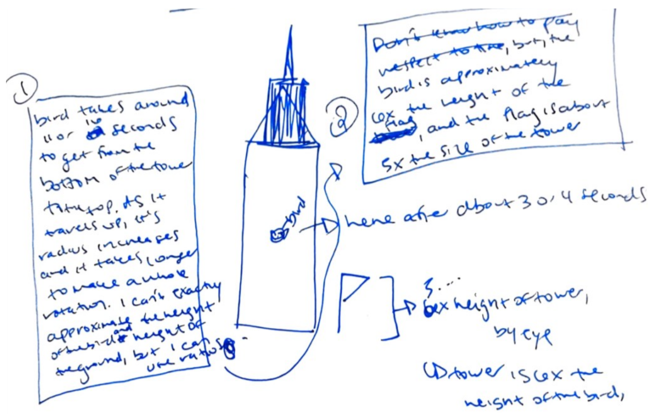

Figure 1: Scanned image showing Jane's Scratchwork for task 1. Transcribed writing:
1. Bird takes around 11 or 16 seconds to get from the bottom of the tower to the top
*2. Flag is approximately 6x height of the tower… tower is 5x height of flag**
3. Tower is 156x height of the bird

Jane used the flag, the tower, and the bird to create a measurement system based on the ratios between each. The flag specifically Jane used as a reference so that they could track the period of each cycle in the bird's movement and identify points to construct her representation. As shown in the next section, Jane specifically used the flag to indicate the side of the tower the bird was on and create "snapshots" of the bird's motion. Between finding a numerical value for the period of the bird's flight and using the flag to create a measurement system of ratios, Jane needed a system of points to create a linear representation.

Creating a Polar Representation

Something to note about Jane's solution to creating a parametric representation for the polar representation is that they misinterpreted the quantities to be coordinated and created a representation that showed the bird's height with the bird's horizontal translation.

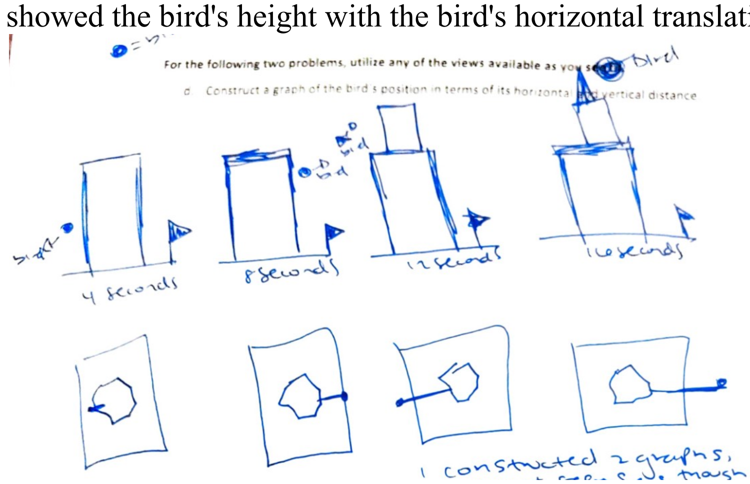

Figure 2: Scanned image showing Jane's Scratchwork for task 2.

This was the closest Jane got to creating a parametric representation, which she accomplished by creating "snapshots". Jane took her pre-existing relations between the bird's height and horizontal position and connected them through time to describe the bird's position. This is where students with a strong process level understanding of covariational reasoning would be able to begin interacting with simultaneous-independent reasoning. Specifically, they could track and understand how the height and the horizontal position relate to each other, with time being implicit in their representation.

In Jane's response there was a clear change-dependent thought process being applied. In the interview Jane explained that she would start with the bird's height at a given time, then determine its horizontal position at that time. In other words, Jane used time to move between each component of the position but could not separate time in the representation. This is where students' definition of *derivative* is a factor since derivatives describe the relationship between different values. Jane described derivatives as, "rates of change that… pay respect to time". Jane's understanding of derivative is tied to how a value relates to time.

Implications

These results show that students with an undeveloped understanding of *function* and *derivative* face challenges applying covariational reasoning. Students with an action level understanding of *function* cannot continually interpret relationships between quantities over extended periods. More research should be directed into understanding the constructions students use to produce continuous parametric representations. This will in turn help students' covariational reasoning abilities because it will prepare students to interact with continuous representations, rather than the "snapshots" we saw in our results.

On the other hand, the role of time in teaching derivatives may need to be de-emphasized. While authors such as Keene (2007) illustrated that students often incorporate time as they consider different attributes of a physical example changing, this may prohibit reasoning about quantities changing irrespective to time. The numerous examples of derivatives with time may encourage students to overgeneralize derivatives as a quantity changing over time. A stronger understanding of derivative would have helped Jane coordinate how the height and the horizontal translation changed relative to each other.

Limitations and Future Work

Despite the results the study was able to produce, there were multiple constraints that appeared. The most immediate was that as a pilot study there was a single participant in the study. This study was also conducted during the COVID-19 pandemic, which meant the study was conducted virtually. This meant that some of the physical actions that students produce when interacting with the instrument were difficult to observe.

After observations during the study and feedback during the interview, the instrument and directions could use further development. For instance, different tools were built in for students to interact with the model which were largely unused. Some of the tools were specifically implemented to determine how students would reason with an invariant relationship, namely whether they would be able to recognize the presence of an invariant relationship and whether they would represent it in their graph. Emphasizing these tools could provide valuable data.

Asides from technical corrections, there are other directions this research could go in future iterations. This target audience for this study were first semester calculus students, which is what led to focusing on students' reasoning capabilities between linear and trigonometric representations since these are some of the first representations students interact with. While could be investigated further, beyond changing the target audience, future iterations could focus on how students are able to use covariational reasonings to interpret relationships with other functions such as exponentials, logarithmic, or even investigating how students recognize and analyze piecewise functions.

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