Developing Proof Comprehension and Proof by Contradiction Through Logical Outlines

Darryl Chamberlain Jr.
University of Florida

Draga Vidakovic
Georgia State University

Proof is central to the curriculum for undergraduate mathematics majors. Despite transition-to-proof courses designed to facilitate the shift from computation-based mathematics to proof-based mathematics, students continue to struggle with mathematical proof. In particular, there are few tasks beyond writing proofs that are specifically designed to develop students' understanding of the proofs they read and the proof methods they utilize. The purpose of this paper is to introduce and discuss the merits of two such tasks: constructing and comparing logical outlines of presented proofs. Grounded in APOS Theory, this paper will illustrate a case study that suggests students can improve their understanding of the proofs they read as well as a particular proof method - proof by contradiction – through these two tasks.

Key words: Proof Comprehension, Proof by Contradiction, Transition-to-proof course, APOS Theory

Proof is central to the curriculum for undergraduate mathematics majors. Despite transition-to-proof courses designed to facilitate the shift from computation-based mathematics to proof-based mathematics, students continue to struggle with mathematical proof (Samkoff & Weber, 2015). Instructors of these courses have stressed that students' ability to understand the proofs they read (*proof comprehension*) is of utmost importance and yet, there are few tasks beyond writing a complete or partial proof of some statement that are designed to improve students' proof comprehension. In short, writing proofs have been the primary tasks used to assess students' understanding of the proofs they read. Noting this, Mejía-Ramos et al. (2012) developed a proof comprehension assessment model that split students' understanding of the proofs they read into two categories: local and holistic. *Local* types of assessment focused on one, or a small number, of statements within a proof whereas *holistic* types of assessment focused on students' understanding of a proof as a whole. Utilizing this assessment model, two groups of researchers developed teaching experiments aimed at improving students' proof comprehension. A brief description of their design and results follows.

Samkoff and Weber (2015) developed a teaching experiment to assess whether certain proof-reading strategies, identified in Weber and Samkoff (2011) and aligned with the previously mentioned proof comprehension assessment model, would aid student understanding. They found that: (1) specific prescriptive guidance helped students implement the strategies more effectively, (2) these strategies were beneficial to students, and (3) that there were impediments to proof comprehension that could not be addressed by these strategies (Samkoff & Weber, 2015). These results suggest that while the proof comprehension model by Mejía-Ramos et al. (2012) may assess student understanding of proof, it cannot, alone, be used as a pedagogical tool to develop instruction for a transition-to-proof course.

Hodds, Alcock, and Inglis (2014) developed a booklet containing self-explanation training focused on the logical relationships within a mathematical proof. Through a series of three experiments, they found that: (1) students who received the self-explanation training scored higher on a comprehension test, (2) self-explanation training increased cognitive engagement with a proof, and (3) a short self-explanation training session within a lecture improved students' proof comprehension and that this comprehension persisted over time (Hodds et al., 2014). These results suggest that focusing on the logical relationships within a mathematical proof can improve students' proof comprehension.

To contribute to the paucity of tasks designed to improve proof comprehension, the authors of this study first utilized APOS Theory to model how students may come to understand the proofs they read and, by extension, how they come to understand a particular proof method: proof by contradiction. These models were then used as a guide to address the following research question:

> *Can outlining given proofs and comparing these outlines enhance students' proof comprehension and overall conception of proof by contradiction?*

The following section briefly describes APOS Theory and the preliminary cognitive model we developed for proof by contradiction to address this research question.

APOS Theory

APOS Theory is a cognitive framework that considers mathematical concepts to be composed of mental Actions, Processes, and Objects that are organized into Schemas. An *Action* is a transformation of Objects by the individual requiring memorized or external, step-by-step instructions on how to perform the operation. As an individual reflects on an Action, he/she can think of these Actions in his/her head without the need to actually perform them based on some memorized facts or external guide; this is referred to as a *Process*. As an individual reflects on a Process, they may think of the Process as a totality and can now perform transformations on the Process; this totality is referred to as an *Object*. Finally, a *Schema* is an individual's collection of Actions, Processes, Objects, and other Schemas that are linked by some general principles to form a coherent framework in the individual's mind (Dubinsky & McDonald, 2001). Utilizing the mental constructs of Actions, Processes, Objects, and Schemas, an outline of the hypothetical constructions students may need to make in order to understand a concept can be developed, referred to as a *genetic decomposition* (Arnon et al., 2014). This genetic decomposition is then used as a foundation to develop instructional materials. A preliminary genetic decomposition for proof by contradiction is provided below.

Preliminary Genetic Decomposition for Proof by Contradiction

1. Action conception of propositional or predicate logic statements as specific step-by-step instructions to construct proofs by contradiction for the following types of statements: (I) implication, (II) non-existence, and (III) uniqueness.
2. Interiorization of each Action in Step 1 individually as general steps to writing a proof by contradiction for statements of the form (I), (II), and (III).
3. Coordination of the Processes from Step 2 into developing a single Process of a proof by contradiction.
4. Encapsulate the Process in Step 3 as an Object by utilizing the law of excluded middle to show proof by contradiction is a valid proof method. Alternatively, encapsulate the Process in Step 3 as an Object by comparing the contradiction proof method to other proof methods.
5. De-encapsulate the Object in Step 4 into a Process similar to Step 3 that then coordinates with a Process conception of other proof methods to prove statements that require two or more proof methods.

In particular for APOS Theory, there is a focus on repeatable transformations that can be reflected on and subsequently generalized by the individual. For proof by contradiction, the repeatable transformation is logically outlining presented proofs (described in Step 1). That is, as students continue to read and reflect on the logical structure of presented proofs (and thus develop their proof comprehension), they can generalize their understanding of these example proofs to develop an internal conception for proof by contradiction based on the structure of the statement proved (described in Step 2). As students encounter different logical structures of proof by contradiction based on the structure of the statement to be proved, they can compare these specific logical structures to develop an internal, general conception for any type of proof by contradiction (described in Step 3). This report will focus on a single student's experience in dealing with tasks designed to induce the mental constructions described by Steps 1, 2, and 3 in the preliminary genetic decomposition. The following section will give an overview of the study's design and a description of the particular tasks this paper will focus on.

Methodology

This report is situated in a larger research project on how students develop an understanding of proof by contradiction within a transition-to-proof course, *Bridge to Higher Mathematics*, at a public R1 university in the southeastern United States. To test the validity of the preliminary genetic decomposition, a five-session teaching experiment was developed and implemented in Fall 2016. These sessions were conducted primarily out-of-class and so the number of sessions a student participated in varied. Of the initial 27 participants, only two completed all five sessions.

This report will focus on two particular tasks developed as part of this teaching experiment: Outlining and Comparing. *Outlining* tasks asked students to logically outline a presented proof by contradiction. These tasks were included to prompt students to identify the logical argument within a presented proof by contradiction. *Comparing* tasks asked students to compare two or more logical outlines of presented proofs. These tasks were used as a reflection tool for students to consider the necessary logical lines of a general proof by contradiction and how these lines logically relate.

Data for this report consists of Yara's responses to these two tasks during the teaching experiment. Yara was a senior Mathematics major with a minor in Educational Psychology. Beyond the required prerequisite courses for *Bridge to Higher Mathematics,* she had already taken *Mathematical Statistics*, *Methods of Regression and Analysis of Variance*, *Foundations of Numbers and Operations*, and *Applied Combinatorics*. However, none of these courses required proof writing and thus *Bridge to Higher Mathematics* was her first experience with formal proofs. She completed all five sessions of the teaching episode. This report focuses on Yara as she was the most elaborative in her responses and provided the most data through which to analyze and support how her understanding of the proofs she read as well as her understanding of proof by contradiction evolved throughout the teaching experiment. The following section will describe how we analyzed her responses.

Data Analysis and Results

All five of Yara's teaching episode sessions were video recorded and then transcribed. Transcripts of these five sessions with Yara were organized and subsequently analyzed using MAXQDA, a qualitative data analysis software. First, sections of the transcripts were grouped by task. Then, Yara's level of understanding proof by contradiction, according to APOS Theory, was analyzed per task. This analysis provided a tool to identify which task or tasks aided her in developing an understanding of the proof method. Due to space constraints, the rest of this section will provide examples from a subset of these sessions on how two tasks, *Outlining* and *Comparing*, aided Yara in developing both a deeper understanding of the proofs she read as well as a more robust understanding of proof by contradiction in general.

Outlining Task

As mentioned previously, *Outlining* tasks asked students to logically outline a presented proof by contradiction. For Outlining task 1, students were given propositional representation for the statement and the majority of lines in the proof. For Outlining task 2, students were given predicate representation for the statement only. Finally, for Outlining tasks 3, 4, and 5, students were not given any logical representation. During these tasks, students were encouraged to use either propositional or predicate symbols to outline the logical structure of the proof. Due to space limitations, this report will focus on Outlining task 3.

The presented proof (Figure 1) and Yara's response to Outlining task 3 (Figure 2) are presented below.

Statement: The equation $5x - 4 = 1$ has a unique solution.
Proof: Assume the equation $5x - 4 = 1$ does not have a unique solution. Then either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. Note $x = 1$ is a solution of $5x - 4 = 1$. Thus there are at least two distinct solutions to the equation $5x - 4 = 1$, call them y and z. As both y and z are solutions of the equation $5x - 4 = 1$, $5y - 4 = 1$ and $5z - 4 = 1$. Then $5y - 4 = 5z - 4$ and so $y = z$. Therefore it is not true that there are at least two distinct solutions to the equation $5x - 4 = 1$. This is a contradiction, as we assumed that either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. Therefore it is not true that the equation $5x - 4 = 1$ does not have a unique solution. In other words, the equation $5x - 4 = 1$ does have a unique solution.

Figure 1: Presented proof for Outlining task 3.

Statement: $\exists!\, x$ s.t. $P(x)$
1. Assume $\sim (\exists!\, x$ s.t. $P(x))$
2. $R \vee Q$
3. $\sim R$
4. Q
5. $5y - 4 = 1 \wedge 5z - 4 = 1$ (Algebra)
6. More algebra ($y = z$)
7. $\sim Q$
8. $\sim (R \vee Q)$
9. $\sim (\sim (\exists!\, x$ s.t. $P(x)))$
10. $\exists!\, x$ s.t. $P(x)$

Figure 2: Yara's logical outline of the presented proof for Outlining task 3.

Yara provided a desired representation of the statement as $(\exists!\, x)(P(x))$ and first line of the outline as $\sim (\exists!\, x)(P(x))$. Then, she switched to propositional logic and initially represented the statement "then either there is no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$" as $P \vee Q$. An excerpt of her thought process behind this representation is provided below.

Yara: And then *[long pause]* and then either there is no solution to the equation or there is at least 2 disinked, I mean, 2 distinct solutions to the equation $5x - 4 = 1$. *[pause]* So it would be the or? Like P or Q?

Teacher: Alright. *[pause for writing]* P or Q. So do these have any relation to the original one?

Yara: No?

Teacher: So if this one doesn't have a relation, then maybe we should call it something else. Like R or Q.

Yara: Oh! To separate that P from that $P(x)$.

Note that she saw the 'or' in the statement and immediately suggested the representation $P \vee Q$. After reflection, she clarified that this P should be changed to separate it from $P(x)$. This suggests the cue word 'or' prompted the representation $P \vee Q$ as a standard representation for an 'or' statement. After teacher's prompting and suggestion to use a different notation, she realized that P should be separate from the initial statement $P(x)$. This suggests that in her initial thinking, Yara automatically used $P \vee Q$, a standard notation for an 'or' statement, without considering the relationship of that statement to the previous statement. In terms of APOS Theory, this excerpt illustrates a possible Action conception of proof by contradiction in relationship to this particular task. However, analyzing further her proof outline, it appears that she is at a higher level of understanding. We illustrate this below.

Overall, her outline contained the two key steps of a proof by contradiction: assuming the negation of the statement is true (line 1) and arriving at a contradiction (line 8). In addition, she verbally described the logical argument of the proof and how lines in the proof related. For example, when she reached the contradiction line in the presented proof, she stated:

> Then... this is a contradiction as we assumed that there is either no solution to the equation $5x - 4 = 1$ or there are at least two distinct solutions to the equation $5x - 4 = 1$. So it would be Q and not Q? Or would we not have to put that because we have it [$R \vee Q$]... It's already labeled out. [...] Okay, so then not... I was just trying to make sure I had it in my head right like, that [$R \vee Q$] would go into not R and not Q.

Her first sentence quoted the line from the presented proof. She then immediately considered the representation $Q \wedge \sim Q$ - the standard representation of a contradiction. Representing a statement by focusing on cue words (i.e., contradiction means $Q \wedge \sim Q$) is indicative of an Action conception of proof by contradiction and suggests, as in the previous paragraph, that Yara did not attend to the logical relation between lines in the proof. However, she then recognized that she would not represent this particular contradiction with $Q \wedge \sim Q$ as she already represented part of this contradiction with $R \vee Q$. Indeed, her final comment "I was just trying to make sure I had it in my head right like, that [$\sim (R \vee Q)$] would go into not R and not Q" suggests that she recognized the logical equivalence $\sim (R \vee Q) \equiv \sim R \wedge \sim Q$ and thus recognized that the contradiction $\sim (R \vee Q) \wedge (R \vee Q)$ was reached. That is, she recognized and verbally described the logical reasoning behind how a contradiction was reached in this particular proof, which is indicative of a Process conception of proof by contradiction. In addition, she generalized lines 5 and 6 in her outline as "algebra" and thus described the purpose of the algebraic manipulations in the overall argument. In other words, Yara was able to use the logical outline to describe the purpose of specific lines in the proof and thus exhibited local comprehension of the presented proof.

Comparing Task

As mentioned previously, *Comparing* tasks asked students to compare two or more logical outlines of presented proofs. These logical outlines were provided by the teacher based on the Outlining tasks. For example, Table 1 illustrates the side-by-side logical outlines from Outlining tasks 1, 2, and 3 that were presented to students for Comparing task 2.

Table 1: Side-by-side logical outlines from Outlining tasks 1, 2, and 3.

Outlining Task 1 **Statement:** $P \to Q$	Outlining Task 2 **Statement:** $(\nexists x)(P(x))$	Outlining Task 3 **Statement:** $(\exists!\, x)(P(x))$
1. Assume $\sim(P \to Q)$ 2. $P \wedge \sim Q$ 3. $\sim Q_k$ 4. $(\sim Q_k \wedge P) \to Q_k$ 5. Q_k 6. $Q_k \wedge \sim Q_k$ 7. $\sim(\sim(P \to Q))$ 8. $P \to Q$	1. Assume $\sim(\nexists x)(P(x))$ 2. $(\exists x)(P(x))$ 3. $P(n)$ 4. Using $P(n)$, get to a contradiction. 5. $\sim(\sim(\nexists x)(P(x)))$ 6. $(\nexists x)(P(x))$	1. Assume $\sim(\exists!\, x)(P(x))$ 2. $\sim(\exists x)(P(x)) \vee (\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$ 3. Show $P(n)$ for some n. 4. $(\exists x, y)(P(x) \wedge P(y) \wedge x \neq y)$ 5. $P(x) \wedge P(y) \to x = y$ 6. $(\nexists x, y)(P(x) \wedge P(y) \wedge x \neq y)$ 7. $\to\leftarrow$ (lines 2, 3, and 6) 8. $\sim(\sim(\exists!\, x)(P(x)))$ 9. $(\exists!\, x)(P(x))$

When prompted to compare the outlines in Table 1, Yara grouped lines together and described a general purpose for each group of lines (see Table 2).

Table 2: Yara's general approach for Comparing tasks 1 and 2.

Comparing Task 1	Comparing Task 2
1. Assume $\sim P$ 2. Rewrite $\sim P$ 3. Look at specific value of step 2. 4. Work (Algebra) 5. Get Contradiction 6. $\sim$ Assumption 7. P	1. Assume $\sim P$ 2. Negate P (Rewrite $\sim P$) 3. Use math skills to get to a contradiction. 4. $\sim$ Assumption 5. P

Yara's approach for proof by contradiction contained both the key steps of a proof by contradiction and descriptions of how these key steps are logically related (e.g., that lines 3 and 7 logically implied line 8). Comparing her general approach between Comparing tasks 1 and 2, we see the she condensed steps 3, 4, and 5 in task 1 into the single step "Use math skills to get to a contradiction." Consider the following exchange as Yara compared the logical outlines from Outlining tasks 1 and 2.

> *Yara*: So I guess it just, maybe it like, depends on the proof, and what you are trying to prove. Whether you do algebra or... umm...
> *Teacher*: So what do we do in that one [*outline during Outlining task 2*]?
> *Yara*: In this one, it says to use $P(x)$, get a contradiction. So we did algebra, right?
> *Teacher*: Yeah, we did algebra that time as well.
> *Yara*: So this one you do... which math skills do you use? Because math skills could mean plenty of things. It could be, like, one of them induction whatever…

From the above excerpt, it is clear that Yara's expression 'math skills' stands for 'mathematical knowledge' since it includes algebraic skills as well as other proof methods such as induction. Yara stated that the steps in the proof depend "... on the proof, and what you are trying to prove." Sometimes, these steps might mean performing some algebra while in the other situation it may mean using a different proof technique. We interpret this to mean that Yara has generalized the notion of a proof by contradiction and exhibited an Object conception of proof by contradiction. She obviously was able to think of these two proof outlines as two entities that could be compared, de-encapsulated each one of them into the processes they came from, and compared separate lines in each outline to distinguish their similarities and differences.

Discussion

Both the Outlining and Comparing tasks enhanced Yara's understanding of the presented proofs in addition to enhancing her understanding of proof by contradiction, as suggested by the preliminary genetic decomposition. These results suggest that the two tasks may be useful in developing transition-to-proof students' proof comprehension as well as their understanding of particular proof methods as they provide a repeatable transformation (outlining the logical structure) that can be reflected on and subsequently generalized by the individual (through comparing logical outlines). While a robust implementation of tasks to transition-to-proof students at a variety of universities would be necessary to validate these tasks, we find these initial results to be encouraging.

Implications for Teaching Practices

This report presented two non-traditional tasks that aided students in developing proof comprehension as well as a robust understanding of proof by contradiction. That is, outlining the logical argument of presented proofs by contradiction (Outlining tasks) and comparing these outlines in order to develop general steps for the proof method (Comparing tasks) differ from the traditional proof writing tasks of "definition-theorem-proof" format transition-to-proof courses (Weber, 2004). This is not to say instructors should abandon proof-writing tasks. Rather, we suggest that Outlining and Comparing tasks should be used in conjunction with traditional proof writing tasks to improve and assess a different aspect of proof: comprehension. The tasks introduced in this report join the tasks based on proof reading strategies by Samkoff and Weber (2015) and the self-explanation training tasks by Hodds et al. (2014) as some of the first tasks designed to improve students' proof comprehension.

Moreover, Outlining and Comparing are the first tasks designed to improve students' comprehension of a particular proof method: proof by contradiction. This is critical as research suggests this method is difficult for students to construct and comprehend (Antonini & Mariotti, 2008; Brown, 2017). These tasks may also provide students a fundamental understanding of the proof method so that other validation tasks, such as critiquing sample proofs and proof editing, may be utilized to further improve on their conception of proof by contradiction.

Finally, these tasks are compatible with other Constructivist frameworks (e.g., Vygotsky's Social Constructivism) and can be used to develop other proof methods (e.g., mathematical induction). Therefore, these two tasks can be used in any transition-to-proof course to develop students' proof comprehension as well as their understanding of particular proof methods.

References

Antonini, S., & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving? *ZDM – The International Journal on Mathematics Education, 40(3),* 401-412.

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. Springer Science & Business Media.

Brown, S. (2017). Are indirect proofs less convincing? A study of students' comparative assessments. *The Journal of Mathematical Behavior.* http://dx.doi.org/10.1016/j.jmathb.2016.12.010

Dubinsky, E. & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In *The Teaching and Learning of Mathematics at University Level* (pp. 275-282). Springer.

Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education*, *45* (1), 62-101.

Mejía-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics, 79*(1), 3-18.

Samkoff, A., & Weber, K. (2015). Lessons learned from an instructional intervention on proof comprehension. *The Journal of Mathematical Behavior, 39,* 28-50.

Weber, K. (2004). A framework for describing the processes that undergraduates use to construct proofs. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 425-432).

Weber K. & Samkoff, A. (2011). Effective strategies that undergraduates use to read and comprehend proofs. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 504-520).