

## Developing Student Understanding: The Case of Proof by Contradiction

Darryl Chamberlain Jr.  
Georgia State University

Draga Vidakovic  
Georgia State University

*Proof is central to the curriculum for undergraduate mathematics majors. Despite transition-to-proof courses designed to facilitate the transition from computation-based mathematics to proof-based mathematics, students continue to struggle with all aspects of mathematical proof. In particular, research suggests that proof by contradiction is an especially difficult proof method for students to construct and comprehend and yet, there are no satisfactory instructional models for how to teach the method. The purpose of this paper is to discuss preliminary results of a teaching experiment on student comprehension of proof by contradiction within a transition-to-proof course. Grounded in APOS Theory, this paper will illustrate that a student's conception of mathematical logic, and quantification in particular, plays an important role in their comprehension of proof by contradiction.*

*Keywords:* Proof by Contradiction, Teaching Experiment, Transition-to-proof course

Proof is central to the curriculum for undergraduate mathematics majors. Indeed, the 2015 *CUPM Curriculum Guide to Majors in the Mathematical Sciences* contains the following recommendation: "Students majoring in the mathematical sciences should learn to read, understand, analyze, and produce proofs at increasing depth as they progress through a major" (Schumacher & Siegel, 2015, p. 11). Despite transition-to-proof courses designed to facilitate the transition from computation-based mathematics to proof-based mathematics, students continue to struggle with all aspects of mathematical proof (Samkoff & Weber, 2015). In particular, research suggests that proof by contradiction is an especially difficult proof method for students to understand and produce (Antonini & Mariotti, 2008; Brown, 2011; Harel & Sowder, 1998). In order to address students' understanding of proof by contradiction, we examined the literature for what proof by contradiction means and what is known about how students develop an understanding of proof by contradiction. A short summary to these questions is provided below.

Proof by contradiction is based on the *law of the excluded middle*: either a statement is true or the negation of the statement is true. By showing the negation of the statement cannot be true (i.e., leads to a contradiction), the statement must be true. However, some scholars do not attribute the law of the excluded middle as the foundation of proof by contradiction. Indeed, Lin, Lee, and Wu Yu (2003) state that "the conceptual knowledge of proof by contradiction is the law of contrapositive" (p. 4-446). Other scholars do not always distinguish between proof by contradiction and proof by contraposition when describing how students understand indirect proofs. For example, Antonini and Mariotti (2008) state that the commonality of negating the thesis in both methods is enough to consider cognitive aspects of students' difficulties with both methods simultaneously. We consider the two indirect proof methods as having distinct underlying concepts: logical equivalence of an implication and its contrapositive (Contraposition) and the law of the excluded middle (Contradiction). These distinct underlying concepts require cognitive models to consider how students develop an understanding of each method separately.

The only current cognitive models of how students understand proof by contradiction have been proposed by Lin, Lee, and Wu Yu (2003) and Antonini and Mariotti (2008). Yet, neither of

these cognitive models are satisfactory as they do not focus on how students develop an understanding of how the law of the excluded middle validates the contradiction proof method. Our larger study aims to address this deficiency in the literature by first modeling how students could come to understand proof by contradiction and then, using this model, develop instructional materials to teach proof by contradiction. The purpose of this paper is to discuss preliminary results on a part of this larger study; that is, to discuss preliminary results of a teaching experiment on a single student's understanding of proof by contradiction. In particular, this paper will address the following question: How does a student's conception of mathematical logic affect their understanding of proof by contradiction? What follows is a detailed discussion of the theoretical framework we will use to answer this question.

### **Theoretical Framework**

Our theoretical framework of choice is APOS Theory – a cognitive framework that considers mathematical concepts to be composed of mental Actions, Processes, and Objects that are organized into Schemas. What follows are definitions of the mental constructions, along with examples within the context of proof by contradiction.

An *Action* is a transformation of mathematical objects by the individual requiring memorized or external step-by-step instructions on how to perform the operation. In the context of proof by contradiction, this may entail a list of steps (either written down or memorized) that a proof by contradiction of an implication statement follows. As an individual reflects on an Action, he/she can think of these Actions in his/her head without the need to actually perform them based on some memorized facts or external guide; this is referred to as a *Process*. Staying within the context of proof by contradiction, a student may interiorize the external list of steps that a particular type of proof by contradiction follows so that these steps become generalized to prove any statement by contradiction. In this way, the student would not need to write the proof in order to talk about what a step may look like in a specific proof. As an individual reflects on a Process, they may think of the Process as a totality and can now perform transformations on the Process; this totality is referred to as an *Object*. Again in the context of proof by contradiction, this could entail comparing proof by contradiction to similar methods such as proof by cases. Finally, a *Schema* is an individual's collection of Actions, Processes, Objects, and other Schemas that are linked by some general principles to form a coherent framework in the individual's mind (Dubinsky & McDonald, 2001). In the case of proof by contradiction, this may include an individual's general proof Schema, mathematical knowledge Schema, and mathematical logic Schema, as well as any steps (external and specific or internal and general) required to write or identify a proof by contradiction.

One of the central goals when using APOS Theory is to develop a series of hypothetical constructions students may need to make in order to understand a concept, referred to as a *genetic decomposition*. This cognitive outline is constructed based on an analysis of the historical development of the concept in question, a literature review of the concept, and the conception of the instructor or researcher. In addition to outlining the mental constructions a student should make in order to understand a concept, a genetic decomposition may include a description of prerequisite knowledge a student should possess in order to begin developing a concept (Arnon et al., 2014). A description of the prerequisite knowledge for proof by contradiction follows.

First, a student should have a general understanding of the propositional logic that underscores all proofs. In particular, a student should be able to perform and understand negations of the following types of statements: without quantification, implications, and single-level quantified statements. This is necessary as every proof by contradiction requires a negation of some statement. In addition to understanding the mathematical logic, a student should be able to move between semantic, symbolic, and algebraic representations of mathematical statements. For example, a student should be able to represent the statement ‘If  $a$  is even, then  $a + 1$  is odd’ as the symbolic implication ‘ $P \rightarrow Q$ ’ as well as the algebraic representation ‘If  $a = 2n$  for  $n \in \mathbb{Z}$ , then  $a + 1 = 2m + 1$  for  $m \in \mathbb{Z}$ ’. Moving between representations of mathematical statements is necessary in any mathematical proof and thus is necessary before developing a notion of proof by contradiction. Finally, a student should be familiar with the direct proof method in a mathematical context as this is the basis of understanding any proof.

With this prerequisite knowledge in mind, we developed a preliminary genetic decomposition. We should be clear that a genetic decomposition need not be unique and so the genetic decomposition below may be one of several different ways in which students may develop an understanding of proof by contradiction.

#### Preliminary Genetic Decomposition for Proof by Contradiction

1. Students outline the propositional logic of a given proof to develop specific, step-by-step instructions to construct proofs by contradiction for the following types of statements: (i) implication, (ii) single-level quantification, and (iii) property claim.
2. Students interiorize each of the Actions in Step 1 into individual Processes by examining the purpose of statements of given proofs. These Processes become general steps to writing a proof by contradiction for statements of the form (i), (ii), and (iii).
3. Students coordinate the Processes from Step 2 by comparing and contrasting the general steps to determine the necessary steps for *any* proof by contradiction. This Process becomes general steps to writing any proof by contradiction and identifying a proof as a proof by contradiction.
4. Students encapsulate the Process in Step 3 as an Object by utilizing the law of excluded middles to show proof by contradiction is a valid proof method. Alternatively, students encapsulate the Process in Step 3 as an Object by comparing proof by contradiction to other, similar proof methods. Students can now comprehend proofs on a holistic level.
5. When necessary, students de-encapsulate the Object in Step 5 into a Process similar to Step 3 that then coordinates with a Process conception of quantification to prove multi-level quantified statements.

First, note that students’ initial conception of proof by contradiction is through the specific steps to construct a proof by contradiction. That is, students think that each of these steps are necessary for a proof to be by contradiction. Students may also need the external cues from specific steps, such as seeing the word “contradiction” near the end of the proof. Students then reflect on these specific steps to considering the general procedure of proof by contradiction. Still, students conceptualize a proof by contradiction as a dynamic procedure that must be followed and thus their conception closely matches the steps to writing a proof by contradiction. It is only after encapsulating the dynamic procedure as a static proof method, by comparing the main idea of the

method to similar methods, that students do not rely on the steps of a proof by contradiction to describe the method.

Once developed, a genetic decomposition is used as a guide to develop all instructional material. One pedagogical approach aligned with APOS Theory is the ACE teaching cycle; an instructional approach that consists of three phases: Activities, Classroom discussion, and Exercises. In the *Activities* phase, students work in groups to complete tasks designed to promote reflective abstraction. These tasks should assist students in making the mental constructions suggested by a genetic decomposition. In the *Classroom discussion* phase, the instructor leads a discussion about the mathematical concepts that the Activities focused on. For example, during this phase the instructor may formally state the theorem that was central in the Action phase and together with students write its complete proof. In the *Exercises* phase, students work on standard problems designed to reinforce the Classroom discussion and support the continued development of the mental constructions suggested by the genetic decomposition. The Exercises also provide students with the opportunity to reinforce and apply what they have learned in the Activities and Classroom discussion phases to related mathematical concepts (Arnon et al., 2014). We utilized the proof comprehension assessment model by Mejía-Ramos et al. (2012) to develop standard proof comprehension questions for the Exercises. As this assessment model was pivotal in the development of instructional materials, we will briefly introduce it below.

Mejía-Ramos et al. (2012) presented a multidimensional model for assessing proof comprehension in undergraduate mathematics. This model contains seven different aspects of proof split into two categories: local and holistic. *Local* types of assessment focus on only one, or a small number, of statements within a proof, whereas *holistic* types of assessment focus on student understanding of a proof as a whole. These seven types of assessment are reproduced below:

1. *Meaning of terms and statements*: items of this type measure students understanding of key terms and statements in the proof;
2. *Logical status of statements and proof framework*: these questions assess students' knowledge of the logical status of statements in the proof and the logical relationship between these statements and the statement being proven;
3. *Justification of claims*: these items address students' comprehension of how each assertion in the proof follows from previous statements in the proof and other proven or assumed statements;
4. *Summarizing via high-level ideas*: these items measure students' grasp of the main idea of the proof and its overarching approach;
5. *Identifying the modular structure*: items of this types address students' comprehension of the proof in terms of its main components/modules and the logical relationship between them;
6. *Identifying the general ideas or methods in another context*: these questions assess students' ability to adapt the ideas and procedures of the proof to solve other proving tasks;
7. *Illustrating with examples*: items of this type measure students' understanding of the proof in terms of its relationship to specific examples. (Mejía-Ramos et al., 2012, p. 15-16).

These assessment questions revolve around the reading and understanding of a presented proof. While the authors caution that all types of assessment questions may not be appropriate for some presented proofs, such as if the proof was too short to warrant breaking into modules, we choose proofs such that each type of assessment question could be applied.

In summary, we used APOS Theory to develop a hypothetical model (genetic decomposition) of possible mental constructions students may need to make in order to develop an understanding of proof by contradiction. We then introduced the ACE teaching cycle, an instructional method closely aligned with APOS Theory, which we used to develop lessons on proof by contradiction. We used the proof comprehension assessment model by Mejía-Ramos et al. (2012) to develop the routine exercise questions for the Exercise phase of the ACE teaching cycle. The methodology section will provide a detailed explanation of the lessons on proof by contradiction as well as provide the context of the study.

## Methodology

In order to assess our preliminary genetic decomposition for proof by contradiction, we conducted a teaching experiment (Steffe & Thompson, 2000). Unlike a typical instructional sequence of the ACE teaching cycle that, in a regular classroom, usually lasts for a week, this teaching experiment consisted of 5 shorter, consecutive teaching sessions each mimicking the ACE teaching cycle. That is, each session consisted of: students working on the Activity worksheet focusing on a particular component of the genetic decomposition for proof by contradiction (A); a discussion about the concepts from the worksheet (C); and a typical series of proof comprehension questions related to the content of the worksheet (E).

Our teaching experiment was conducted at a large, public R1 university in the southeastern United States with students from a transition-to-proof course. This course, *Bridge to Higher Mathematics*, is the first course in which students at this university are formally introduced to mathematical proofs and their accompanying methods of proof. *Bridge to Higher Mathematics* has the following general learning outcomes:

- Basic Logic (e.g. truth tables, negation, quantification),
- Proof Methods (e.g., direct, contradiction, induction),
- Introductory Set Theory (e.g., union, intersection, power set, cardinality), and
- Introductory Analysis (e.g., least upper bound and greatest lower bound, open/closed sets, limit points).

We began with teaching episodes two weeks into the course, just after a review of basic logic and an introduction to direct proofs. *Bridge to Higher Mathematics* is a primarily lecture-based course where proof and proof methods are taught primarily through proof construction (i.e., asking students to replicate proofs). Thus, in both teaching style and assessment, the designed teaching experiment deviated from the regular classroom. However, the teaching episodes did correspond to the content and proofs that were normally covered in the course.

Data for this report was collected during Summer 2016 from teaching episodes with Chandler. Chandler was an untraditional student: he had already graduated once with a bachelor degree (in an unspecified topic) and was unsure as to whether or not he wanted to complete a bachelor's degree in mathematics. As a senior, Chandler had already taken courses beyond the

prerequisites for *Bridge to Higher Mathematics*, including *Differential Equations*, *Probability Theory*, and *Vector Calculus*. However, none of these additional mathematics courses required proof writing and thus *Bridge to Higher Mathematics* was still Chandler's first experience learning how to read and write formal proofs. While Chandler initially consented to completing all five teaching episodes, he dropped the course before mid-semester. As such, Chandler only completed the first three teaching episodes.

We choose to focus on Chandler for this preliminary report for two primary reasons. First and foremost, Chandler's progression of understanding proof by contradiction is similar to the majority of other participants and thus can be considered representative of a general participant's understanding. In addition, Chandler was open to communicating, answering, and asking questions without any reservation. This provided us with rich data on his thought process. The next section will describe how we analyzed Chandler's thought process through the three teaching episodes he completed.

### **Data Analysis**

All three teaching sessions were video recorded and then transcribed by the teacher/researcher. Transcripts of these three sessions went through multiple passes of analysis. First, we identified Chandler's conception of proof by contradiction, according to APOS Theory, for each teaching episode. Next, we identified Chandler's local and holistic understanding of proof by contradiction and proof in general, according to the proof comprehension assessment model by Mejia Ramos et al. (2012), during each teaching episode. We then open coded for other themes that affected Chandler's understanding of proof by contradiction, during which quantification emerged. Finally, we examined quantification's role in Chandler's understanding per teaching episode. What follows are the results from this analysis.

### **Results**

For each episode, we will first present the statement and proof Chandler was asked to read as well as provide the comprehension questions he was then asked to complete. Afterwards, we will provide some of Chandler's responses that best illustrate his understanding of proof by contradiction in general and his understanding of the presented proof. When applicable, we will include a discussion of how mathematical logic, and quantification in particular, affected Chandler's proof comprehension.

#### **Pre-Teaching Episodes**

Before the first teaching episode, we asked Chandler a few basic questions to ascertain his background with mathematical proof. In particular, we were focused on determining whether Chandler possessed the following prerequisite knowledge necessary to begin developing a conception of proof by contradiction: mathematical logic (especially negating statements), how to transition between representations, and proof in general. Chandler exhibited at least an Action conception of mathematical logic as he was able to describe rules to use in order to negate propositional statements. Chandler was also able to answer questions that required him to transition between mathematical statements, their propositional representations, and their algebraic representations. Finally, with respect to his understanding of proof in general, Chandler gave the following definition:

*An effort to show something is true or false based on real evidence and facts.*

We see his definition as a dynamic procedure, i.e. an effort to show something, and so infer Chandler possesses a Process conception of proof. Based on his responses, we believe Chandler had the prerequisite knowledge to begin developing a conception of proof by contradiction.

We also wanted to know what conception, if any, Chandler had of proof by contradiction before the first teaching episode. When asked for a definition of proof by contradiction, he stated:

*Demonstrating the opposite of a statement is false, so the statement is true.*

Note there is no mention of an assumption or contradiction in terms of “demonstrating the opposite of a statement is false” and thus this is not a complete definition of proof by contradiction. At most, this is a definition for a general type of indirect proof that utilizes the tautological equivalence of a double-negated statement and the statement itself. We therefore do not believe Chandler has a conception of a mathematical proof by contradiction.

### **Teaching Episode 1**

Teaching episode 1 began with Chandler converting a series of statements into propositional and predicate logic on his own. After checking his answers to these conversions, it was clear that Chandler was able to convert mathematical statements with a single quantifier into a logical form. Chandler was then asked to read the following statement and proof.

*Statement 1:* The set of primes is infinite.

*Proof 1:* Suppose the set of primes is finite. Let  $p_1, p_2, p_3, \dots, p_k$  be all those primes with  $p_1 < p_2 < p_3 < \dots < p_k$ . Let  $n$  be one more than the product of all of them. That is,  $n = (p_1 p_2 p_3 \dots p_k) + 1$ . Then  $n$  is a natural number greater than 1, so  $n$  has a prime divisor  $q$ . Since  $q$  is prime,  $q > 1$ . Since  $q$  is prime and  $p_1, p_2, p_3, \dots, p_k$  are all of the primes,  $q$  is one of the  $p_i$  in the list. Thus,  $q$  divides the product  $p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_k$ . Since  $q$  divides  $n$ ,  $q$  divides the difference  $n - (p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_k)$ . But this difference is 1, so  $q = 1$ . From the contradiction,  $q > 1$  and  $q = 1$ , we conclude that the assumption that the set of primes is finite is false. Therefore, the set of primes is infinite.

Chandler was then prompted to answer the following comprehension questions on his own.

1. Please give an example of a set that is infinite and explain why it is infinite.
2. Why does  $n$  have to have a prime divisor?
3. Why exactly can one conclude that  $q$  divides the difference  $n - (p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_k)$ ?
4. What is the purpose of the statement “Let  $p_1, p_2, p_3, \dots, p_k$  be all those primes with  $p_1 < p_2 < p_3 < \dots < p_k$ .”?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of the proof?
7. In the proof, we define  $n = (p_1 p_2 p_3 \dots p_k) + 1$ . Would the proof still work if we instead defined  $n = (p_1 p_2 p_3 \dots p_k) + 31$ ? Why or why not?
8. Define the set  $S_k = \{2, 3, 4, \dots, k\}$  for any  $k > 2$ . Using the method of this proof, show that for any  $k > 2$ , there exists a natural number greater than 1 that is not divisible by any element in  $S_k$ .

Considerable attention was given to local comprehension questions 1-4 as Chandler was not able to discuss questions 5-8. We highlight his most enlightening responses to the Activity and these comprehension questions below.

After reading the statement and its proof, Chandler was asked to describe or define proof by contradiction. He said:

*Aren't you supposed to show not the conclusion, or try to prove not the conclusion and try to come up with something that is?*

While this definition discusses the final steps of a proof by contradiction, it focuses on “showing not the conclusion” rather than discussing how the initial assumption and contradiction relate to the original statement. Thus, we believe Chandler exhibited a PreAction conception of proof by contradiction.

Chandler was able to describe the meaning of key words and phrases, such as when he described infinity as:

*Every time it [a real number] goes up, you can still go up by 1.*

However, Chandler was unable to describe the purpose of statements and provide justification for lines in the proof. For example, he initially wrote that the purpose of the statement ‘Let  $p_1, p_2, p_3, \dots, p_k$  be all those primes with  $p_1 < p_2 < p_3 < \dots < p_k$ ’ was:

*To construct an infinite number of possibilities for number  $n$ , since numbers are composites of primes  $p$  and  $n$  is not a particular natural number.*

When the teacher asked Chandler to clarify his written response during the discussion, he said:

*Infinity. Uh, well the set of primes is infinite. [long pause] Is it,  $n$  is composed of a finite number of parts?*

In both cases, Chandler did not recognize the statement as defining the assumed finite set of primes so that they may be used to produce a new prime. When the teacher asked Chandler to provide justification for the line ‘ $q$  divides the difference  $n - (p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_k)$ ’, Chandler realized that he wrote multiple answers and even expressed confusion on what he was thinking:

*I wonder what I decided... there's several things boxed in.*

All of the answers Chandler initially provided involved rewriting statements in the proof without further clarification. Even after an extended discussion considering the statement with a small number of primes, Chandler was unable to provide justification for the line. These examples suggest that while Chandler could understand what individual statements meant, he did not know how these statements logically or mathematically followed from one another. Thus, we believe Chandler showed some local understanding of the presented proof.

When asked to summarize the main idea of the proof, Chandler restated the statement. When asked what the key steps of the proof were, he said:

*Supposing the contradiction is true, that the set of primes is finite and showing how to prove that by reducing an infinite number of primes down to the number 1... I don't know what that means.*



When asked whether the proof would still work if we changed  $n = (p_1 p_2 p_3 \dots p_k) + 1$  to  $n = (p_1 p_2 p_3 \dots p_k) + 31$ , he said that it would “Because the difference is 31, so  $q$  is 31 and  $q$  is still greater than 1. I don’t know.” In the questions above, Chandler does not exhibit any holistic understanding. Yet, Chandler does show some holistic understanding in his attempt to show that for any  $k > 2$ , there exists a natural number greater than 1 that is not divisible by any element in  $S_k = \{2, 3, 4, \dots, k\}$ . Chandler was able to identify the previous construction of  $n$  as being necessary for the new proof and began his proof by appropriately modifying the new number  $n = (2 \cdot 3 \cdot 4 \cdot \dots \cdot k) + 1$ . Rather than showing that this number is not divisible by any element in  $S_k$ , Chandler tried to mimic the presented proof. It is likely that Chandler was unsuccessful in writing the proof because he did not understand the multi-level quantified statement and, in particular, did not know how to negate this statement. Consider the following discussion between Chandler and the teacher about his proof:

*T: Any idea how to negate this statement?*

*C: And I thought of that but I just, it didn’t seem following along, and I sure didn’t [know] how to get it following along like this one. I think the contradiction would be: for all natural numbers greater than 1, wait [pause, mumbling to self] for all natural numbers greater than 1... I don’t know.*

*T: Okay, let’s try to rewrite this in symbols, maybe.*

*[After prompting, Chandler writes  $\forall k > 2, \exists n > 1 \in \mathbb{N} \text{ s.t. } s \nmid n \ \forall s \in S_k$ ]*

*T: Can you negate this now?*

*C: I think, piece by piece. There’s a whole string of these, I don’t know. I think for all  $n$  less than or equal to 1, or yeah, is that right?*

*T: We also do the, for every  $k$  greater than 2, so maybe it’s “there exists a  $k$  greater than 2” so there is actually one of them.*

*C: Or, wait, which part? You don’t negate this [points at  $n > 1$ ], you negate this [points at  $\forall$ ]. The quantifier?*

*T: Yes, yes.*

*C: So you don’t negate the [trails off].*

*[After more discussion on negating parts of the statement.]*

*C: I’m trying to remember [pause] such that for some  $s$ ,  $s$  divides  $n$ , for  $s$  in the set. Is that the entirety?*

While Chandler was unable to negate the multi-quantified statement (either as a mathematical statement or in algebraic notation), he did say

*And I thought of that [negating the statement] but I just, it didn’t seem following along, and I sure didn’t how to get it following along like this one...*

which indicates that he may have been able to complete proof had he been able to negate the multi-quantified statement. We also see phrases such as “I’m trying to remember...” and errors such as “I think [the negation would be] for all  $n$  less than or equal to 1...” that indicate Chandler may need an external guide or table to negate quantified statements even single-quantified statements. At least in this case, it is possible that Chandler’s conception of quantification was inhibiting his holistic comprehension of the proof.

## Teaching Episode 2

Teaching episode 2 began with Chandler examining the following statement and proof, after which he was asked to provide a definition of proof by contradiction based on the presented proof.

*Statement 2:* If every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

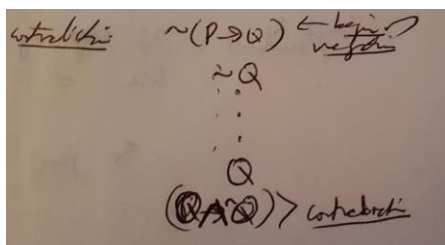
*Proof 2:* Assume that every even natural number greater than 2 is the sum of two primes and that it is not the case that every odd natural number greater than 5 is the sum of three primes. Then there exists an odd natural number greater than 5 that is not the sum of three primes, call it  $k$ . Then  $k = 2n + 1$ . Since  $k > 5$ ,  $2n + 1 > 5$  and so  $n > 2$ . Rewriting  $k$ , we have  $k = 2(n - 1) + 3$ . Since  $n > 2$ ,  $2(n - 1) > 2$  and so is the sum of two primes:  $p$  and  $q$ . Thus  $k = p + q + 3$ . This is a contradiction as we assumed  $k$  was not the sum of three primes. Therefore if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

After a discussion on the underlying structure of the presented proof and how this related to proof by contradiction in general, Chandler was asked to answer the following comprehension questions on his own.

1. Please give an example of a prime number and explain why it is prime.
2. Why is every even natural number greater than 2 the sum of two primes?
3. Why exactly can one conclude that  $k - 3$  is the sum of two primes?
4. What is the purpose of the statement “Since  $k > 5$ ,  $2n + 1 > 5$  and so  $n > 2$ .”?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of the proof?
7. In the proof, we rewrite  $k = 2(n - 1) + 3$ . Would the proof still work if we instead rewrite  $k = 2(n - 2) + 5$ ? Why or why not?
8. Using the method of this proof, show that: if every odd natural number greater than 5 is the sum of three primes and 3 is one of those primes, then every even natural number greater than 2 is the sum of two primes.

Unlike teaching episode 1, Chandler was able to talk more about the holistic questions than he could previously. We highlight his most enlightening responses below.

After discussing the specific structure of the presented proof, Chandler came up with the following step-by-step guide for a general implication statement:



<u>Contradiction</u>	$\sim (P \rightarrow Q)$	$\leftarrow$ <u>begin negation</u>
	$\sim Q$	
	$\vdots$	
	$Q$	
	$(Q \wedge \sim Q)$	$>$ <u>contradiction</u>

Figure 1: Chandler’s written structure (left) and his recreated structure (right)

First, note that the logical connection between the assumption,  $\sim (P \rightarrow Q)$ , and the next line,  $\sim Q$ , is omitted as well as the logical connection between the assumption and contradiction in order to conclude the proof. Also, there is no mention of the role of  $P$  in the proof. These omissions indicate that the structure above is meant to *write* a complete proof by contradiction and not to outline the logical structure of the method. In other words, Chandler focus is on the procedure and not in understanding why the procedure works. Thus, we believe he exhibits an Action conception of proof by contradiction.

After reading the first line of the proof and before attempting any of the comprehension questions, Chandler said:

*I'm not sure when something contradicts, if you begin with 1, step 1. The assumption, or the given, was that  $P$  and [sic]  $Q$ . If you begin with  $P$  and not  $Q$ , it's already a contradiction, isn't it? No?*

Once the teacher explained the logical equivalence between the hidden initial assumption step,  $\sim (P \rightarrow Q)$ , and the first statement of the proof,  $P \wedge \sim Q$ , Chandler was able to understand the logical relation between the statement and the first line of the proof. However, rather than include this equivalence in his proof by contradiction steps above, he only included  $\sim Q$ , the consequent of  $P \wedge \sim Q$  the proof uses to move forward and end up with a contradiction. While his steps may now avoid attending to local understanding of the statement and assumption, we see an instance in which Chandler, unprompted, attended to the local understanding of a particular proof by examining the propositional logic.

When writing his proof for comprehension question 8, Chandler had difficulty negating statement  $Q$  (a key step in his current conception for proof by contradiction) without first translating the statement into propositional logic and using rules to negate the propositional logic. An excerpt of what transpired is provided below:

*T: Do you have any ideas on how to approach this proof? [Long pause]*

*C: Not  $Q$  would be 'every even natural number greater than 2 is not the sum of two primes'? No, all? I don't know.*

*[Teacher prompts Chandler to write the statement in propositional logic]*

*T: Alright. So for every  $n$  here, that's how you said it to me. So parentheses for all  $n$ , in the natural numbers, if  $n$  is greater than 2, then  $n$  is  $p + q$ . So what would be the negation of this statement?*

*C: There is a natural number...*

*T: There is a natural number.*

*C:  $n$  greater than 2 that is not equal to  $p + q$ .*

We see Chandler utilizing his general procedure for implication statements by starting with negating statement  $Q$ . Once he wrote the statement in propositional logic, Chandler was able to recognize the negation of a 'for all' statement as 'there exists' and continue using his procedure to complete the proof. This suggests that Chandler's conception of quantification still inhibited his holistic understanding of the proof.

### Teaching Episode 3

Teaching episode 3 began with Chandler examining the following statement and proof, after which he was asked to provide a definition of proof by contradiction based on the presented proof.

*Statement 3:* There is no odd integer that can be expressed in the form  $4j - 1$  and in the form  $4k + 1$  for integers  $j$  and  $k$ .

*Proof 3:* Assume it is not true that there is no odd integer that can be expressed in the form  $4j - 1$  and in the form  $4k + 1$  for integers  $j$  and  $k$ . Then there is an odd integer that can be expressed in the form  $4j - 1$  and in the form  $4k + 1$  for integers  $j$  and  $k$ . Let  $n$  be that integer; that is,  $\exists n \in \mathbb{N}$  such that  $n = 4j - 1$  and  $n = 4k + 1$  for  $j, k \in \mathbb{Z}$ . Then  $4j - 1 = 4k + 1$  and so  $2j = 2k + 1$ . Note that  $2j$  is an even number and, since  $2j = 2k + 1$ ,  $2j$  is an odd number. A number cannot be both even and odd and thus this is a contradiction.

Therefore, it is not true that it is not true that there is no odd integer that can be expressed in the form  $4j - 1$  and in the form  $4k + 1$  for integers  $j$  and  $k$ . In other words, there is no odd integer that can be expressed in the form  $4j - 1$  and in the form  $4k + 1$  for integers  $j$  and  $k$ .

Chandler was then asked to compare and contrast Proof 2 and Proof 3. After a discussion on the similarities and differences of the underlying structures of these proofs how this related to proof by contradiction in general, Chandler was asked to answer the following comprehension questions on his own.

1. Please give an example of an integer that is odd and explain why it is odd.
2. What kinds of numbers can be expressed in the form  $4j - 1$ ?
3. Why exactly can we assume “ $\exists n \in \mathbb{N}$  such that  $n = 4j - 1$  and  $n = 4k + 1$  for  $j, k \in \mathbb{Z}$ .”?
4. What is the purpose of the statement “Note that  $2j$  is an even number and, since  $2j = 2k + 1$ ,  $2j$  is an odd number.”?
5. Summarize in your own words the main idea of this proof.
6. What do you think are the key steps of the proof?
7. In the statement, we have  $4j - 1$  and  $4k + 1$ . Would the proof still work if we instead have  $4j - 3$  and  $4k + 3$ ? Why or why not?
8. Using the method of this proof, show that there is no odd integer than can be expressed in the form  $8j - 1$  and  $8k + 1$  for integers  $j$  and  $k$ .

The teacher/research focused Chandler's attention on the holistic comprehension questions as Chandler was able to give quick, confident answers to the local comprehension questions. We provide some of his most enlightening responses below.

As mentioned above, Chandler was asked to compare Proof 2 and Proof 3 in their presented form. An excerpt of this comparison is provided below, along with a recreation of Chandler's two proof structures that he eventually compared in place of the semantic representations of the proofs.

*C:* They are both contradiction, proof by contradiction, and [pause] I don't know.

*T:* Well, maybe... let's see. You have the structures of both proofs, right?

[Teacher prompts Chandler to examine the structures he already developed.]

*T: So you were saying something before you started talking about the  $\sim Q$ ; you were pointing at [steps] 5-7.*

*C: Yeah. Those are similar – [steps] 4-7 are similar to this, [steps] 4-6, of course.*

Table 2: Recreation of Chandler's proof structures for implication (left) and nonexistence (right) statements.

$P \rightarrow Q$	$(\nexists x)(P(x))$
1. $P \wedge \sim Q$	1. Assume $\sim (\exists x)(P(x))$
2. $\sim Q$	2. $\sim (\exists x)(P(x))$ [sic]
3. $\sim Q \rightarrow R_1$	3. $P(n)$
4. $R_1 \rightarrow R_2$	4. Algebra until contradiction: $4j - 1 = 4k + 1$ means $2j = 2k + 1$
5. $R_2 \rightarrow R_3$	5. $\sim (\exists x)P(x)$
6. $R_3 \rightarrow Q$	6. $(\exists x)P(x)$
7. $Q \wedge \sim Q$	
8. $\sim (P \wedge \sim Q) = P \rightarrow Q$	

What Chandler is talking about here is the role of parts of the proof. In particular, he is considering how each structure arrives at the contradiction and uses this contradiction to complete the proof (Note: although he initially excludes step 8 in the implication structure as seen in Table 2, he clarifies the role of this step later in the discussion). Chandler uses this insight when he developed a general set of steps for any kind of proof by contradiction, replicated below:

1. *Assume the claim is false.*
2. *Rewrite the assumption.*
3. *Do algebra until contradiction.*
4. *The assumption is false, so the original claim is true.*

Chandler leveraged the propositional and quantified logical structure of the proofs to produce a general, non-logical list of steps for *any* proof by contradiction and thus we believe he developed a Process conception of proof by contradiction.

Chandler used of the general steps above to identify the purpose of single statements in the presented proof. When asked why the proof can assume ‘there is an odd integer  $n$  such that  $n = 4j - 1$  and  $n = 4k + 1$  for  $j, k \in \mathbb{Z}$ ’, Chandler responded:

*Because we are claiming the negation of “there is no odd integer  $n$  such that  $n = 4j - 1$  and  $n = 4k + 1$ ” which is “there is an odd integer  $n$ ...”*

We see that Chandler recognized ‘there is an odd integer  $n$ ...’ as a rewrite of the assumption without needing to refer to the logical equivalence of the statements. In addition, Chandler identified that the purpose of the statement ‘Note that  $2j$  is an even number and, since  $2j = 2k + 1$ ,  $2j$  is an odd number’ was to identify the contradiction. In both cases, we see Chandler relating the logical status and justification of statements to the general steps he has developed for a proof by contradiction. In doing so, he no longer needed to refer to the underlying structure of the proof in order to locally comprehend the presented proof.

Unlike teaching episodes 1 and 2, Chandler was able to describe the key steps of the presented proof and stated:

*C: Rewording it, I think. Rewording the claim into a negation and the algebra.*

*T: That makes sense. Alright.*

*C: For this one. I mean... I don't know.*

What Chandler is describing as the key steps for this proof are steps 2 and 3 for general proofs by contradiction that he constructed during the Discussion of this teaching episode, though he clarifies that these may only be the key steps for this proof. Based on the previous steps he has listed as key steps, it seems that Chandler conceptualized the key steps of a proof as the steps in which he found to be most difficult. With this in mind, his clarification that the key steps of another proof may be different likely refers to the idea that the *difficult steps* of another proof may be different and not that the *general steps* of another proof by contradiction may be different. Thus, we believe Chandler exhibits holistic understanding of the presented proof based on his general understanding of proof by contradiction.

### **Discussion**

It was evident that logic, and quantification in particular, played an important role in Chandler's understanding of particular proofs and proof by contradiction in general. First, he leveraged propositional logic to develop specific steps for a particular kind of proof by contradiction as well as considered the logical connection between the statement and assumption of a particular proof. We also observed that Chandler's initial difficulties with negating quantified statements inhibited his ability to transfer the methods of a particular proof into a similar context. By the end of the teaching experiment, Chandler developed a list of general steps any proof by contradiction should contain through the comparison of two different logical structures of particular proofs by contradiction. These steps not only aided Chandler in transferring the method of one proof to similar proof, but they aided Chandler in recognizing the logical status and justification of statements in a particular proof.

Chandler's conception of mathematical logic seemed to, at times, inhibit his proof comprehension. In particular, Chandler exhibited a need to convert statements into propositional or predicate logic in order to either negate the statement or recognize the logical relation between pairs of statements. This is especially important to highlight for proof by contradiction as students are not convinced of the validity of the proof method (Brown, 2011). However, by exploring the logical relation between the assumption, contradiction, and original statement, students may become convinced and thus more accepting of the method. At the very least, we found that prompting Chandler to explore the logical relation between statements in a proof and to develop key steps aided him in developing a robust understanding of proof by contradiction.

### **Future Plans**

Overall, the teaching experiment seemed to aid Chandler in developing a robust understanding of proof by contradiction by focusing on the logical relationships within proofs. These results further a growing belief that attending to the logical relationship of statements may greatly improve proof comprehension (Brown, 2013; Hodds et al., 2014). Before we consider the

teaching experiment useful to a general student at the university, we need to consider more case studies. Five more students from *Bridge to Higher Mathematics* volunteered and participated in at least 3 teaching episodes during Fall 2016. We plan to analyze these students' understanding of proof by contradiction in terms of APOS Theory and their understanding of specific proofs via the proof comprehension assessment model by Mejía-Ramos et al. (2012) in a similar fashion to Chandler. Positive results from these cases would strengthen our claim that the teaching experiment would be useful for general transition-to-proof students and allow us to examine the role of mathematical logic in understanding proof by contradiction.

### References

- Antonini, S., & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving? *ZDM – The International Journal on Mathematics Education*, 40 (3), 401-412.
- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. Springer Science & Business Media.
- Brown, S. (2011). An investigation of students proof preferences: The case of indirect proofs. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 14<sup>th</sup> Annual Conference on Research in Undergraduate Mathematics Education* (Vol. 4, pp. 31-36).
- Brown, S. (2013). Partial unpacking and indirect proofs: A student of students' productive use of the symbolic proof scheme." *Proceedings of the 16<sup>th</sup> Annual Conference on Research in Undergraduate Mathematics Education* (Vol. 2, pp. 47-54).
- Dubinsky, E. & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In *The teaching and Learning of Mathematics at University Level* (pp. 275-282). Springer.
- Harel, G. & Sowder, L. (1998). Students proof schemes: Results from exploratory studies. *Research in collegiate mathematics education III*, 234-283.
- Lin, F.-L., Lee, Y.-S., & Wu Yu, J.-Y. (2003). Students' understanding of proof by contradiction. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA* (Vol. 4, pp. 443-449).
- Mejía-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, 79(1), 3-18.
- Samkoff, A., & Weber, K. (2015). Lessons learned from an instructional intervention on proof comprehension. *The Journal of Mathematical Behavior*, 39, 28-50.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*, 267-306.