Patterns on liquid surfaces: cnoidal waves, compactons and scaling

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Abstract

Localized patterns and nonlinear oscillation formation on the bounded free surface of an ideal incompressible liquid are analytically investigated . Cnoidal modes, solitons and compactons, as traveling non-axially symmetric shapes are discused. A finite-difference differential generalized Korteweg-de Vries equation is shown to describe the three-dimensional motion of the fluid surface and the limit of long and shallow channels one reobtains the well known KdV equation. A tentative expansion formula for the representation of the general solution of a nonlinear equation, for given initial condition is introduced on a graphical-algebraic basis. The model is useful in multilayer fluid dynamics, cluster formation, and nuclear physics since, up to an overall scale, these systems display liquid free surface behavior.

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1 Introduction

Liquid oscillations on bounded surfaces have been studied intensively, both theoretically [1-3] and experimentally [4-6]. The small-amplitude oscillations of incompressible drops maintained by surface tension are usually characterized by their fundamental linear modes of motion in terms of spherical harmonics [1-3]. Nonlinear oscillations of a liquid drop introduce new phenomena and more complicated patterns (higher resonances, solitons, compactons, breakup and fragmentation, fractal structures, superdeformed shapes) than can be described by a linear theory. Nonlinearities in the description of an ideal drop demonstrating irrotational flow arise from Bernoulli's equation for the pressure field and from the kinematic surface boundary conditions [7]. Computer simulations have been carried for non-linear axial oscillations and they are in very good agreement with experiments [4-6].

The majority of experiments show a rich variety of complicated shapes, many related to the spinning, breaking, fission and fusion of liquid drops. There are experiments [6] and numerical simulations [2] where special rotational patterns of circulation emerge: a running wave originates on the surface of the drop and then propagates inward. Recent results (superconductors [8], catalytic patterns [9], quasi-molecular spectra [10], numerical tests on higher order non-linear equations [11] and analytical calculations on the non-compact real axis [12-13]) show shape-stable traveling waves for nonlinear systems with compact geometry. Recent studies showed that a similar one-dimensional analysis for the process of cluster emission from heavy nuclei and quasi-molecular spectra of nuclear molecules yields good agreement with experiment [10]. Such solutions are stable and express to a good extent the formation and stability of patterns, clusters, droplets, etc. However, even localised, they have nor compact support neither periodicity (excepting some intermediate steps of the cnoidal solutions, [10,13]), creating thus difficulties when analysing on compact surfaces.

In the present paper we comment on the cnoidal-towards-solitons solution investigated in [9,12], especially from the energy point of view. We introduce here a new nonlinear 3-dimensional dynamical model of the surface, in compact geometry (pools, droplets, bubbles, shells), inspired by [12], and we investigate the possibilities to obtain compacton-like solutions for this model. We also study the scale symmetries of such solutions.

The model in [9,12] consider the nolinear hydrodynamic equations of the surface of a liquid drop and show their direct connection to KdV or MKdV systems. Traveling solutions that are cnoidal waves are obtained [10,13] and they generate multiscale patterns ranging from small harmonic oscillations (linearized model), to nonlinear oscillations, up to solitary waves. These non-axis-symmetric localized shapes are described by a KdV Hamiltonian system, too, which results as the second order approximation of the general Hamiltonian, next corrextion from the linear harmonic shape oscillations. Such rotons were observed experimentally when the shape oscillations of a droplet became nonlinear [4,6,8,13].

2 Liquid drop cnoidal and soliton solutions from Hamiltonian approach

The dynamics governing one-dimensional surface oscillations of a perfect (ρ =const.), irrotational fluid drop (or bubble, shell) can be described by the velocity field Φ and a corresponding Hamiltonian [1-3,7,10,13]. By expanding the Hamiltonian and dynamical equations in terms of a small parameter, i.e. the amplitude of the perturbation η over the radius of drop R_0, the usual linear theory is recovered in the first order. Higher order non-linear terms introduce deviations and produce large surface oscillations like cnoidal waves [7]. These oscillations, under conditions of a rigid core of radius $R_0 - h$ and non-zero angular momentum, transform into solitary waves. In the following, by using the calculation developed in [10], we present the Hamiltonian approach for the liquid drops nonlinear oscillations. However, this approach is different from the nuclear liquid drop model point of view in [10], since we do not use here the nuclear interaction (shell corrections) responsible for the formation of different potential valleys.

The total hydrodynamic energy E consists of the sum of the kinetic T and potential U energies of the liquid drop. The shape function is assumed to factorize, $r(\theta, \phi, t) = R_0(1 + g(\theta)\eta(\phi, t))$. All terms that depend on θ are absorbed in the coefficients of some integrals and the energy reduces to a functional of η only. The potential energy is given by the surface energy $U_S = \sigma(\mathcal{A}_\eta - \mathcal{A}_0)|_{V_0}$, where σ is the surface pressure coefficient, $\mathcal{A}_\eta$ is the area of the deformed drop, and $\mathcal{A}_0$ the area of the spherical drop, of constant volume V_0. The kinetic energy $T = \rho \oint_\Sigma \Phi \nabla \Phi \cdot d\vec{S}/2$, [1-3,10,13], the kinematic free surface boundary condition $\Phi_r = \partial_t r + (\partial_\theta r)\Phi_\theta/r^2 + (\partial_\phi r)\Phi_\phi/r^2 \sin\theta$, and the boundary condition for the radial velocity on the inner surface $\partial_r \Phi|_{r=R_0-h} = 0$, [7], result in the expression [2,3,10]

$$T = \frac{R_0^2 \rho}{2} \int_0^\pi \int_0^{2\pi} \frac{R_0 \Phi \eta_t \sin\theta + \frac{1}{R_0} g \eta_\phi \Phi \Phi_\phi (1 - \sin\theta)}{\sqrt{1 + g_\theta^2 \eta^2 + g^2 \eta_\phi^2}} d\theta d\phi. \tag{1}$$

If the total energy, written in the second order in η, is taken to be a Hamiltonian $H[\eta]$, the time derivative of any quantity $F[\eta]$ is given by $F_t = [F, H]$. Defining $F = \int_0^{2\pi} \eta(\phi - Vt) d\phi$ it results ([10], last reference)

$$\frac{dF}{dt} = \int_0^{2\pi} \eta_t d\phi = \int_0^{2\pi} (2C_2 \eta_\phi + 6C_3 \eta \eta_\phi - 2C_4 \eta_{\phi\phi\phi}) d\phi = 0, \tag{2}$$

which leads to the KdV equation. Here $\mathcal{C}_2 = \sigma R_0^2 (S_{1,0}^{1,0} + S_{0,1}^{1,0}/2) + R_0^6 \rho V^2 C_{2,-1}^{3,-1}/2$, $\mathcal{C}_3 = \sigma R_0^2 S_{1,2}^{1,0}/2 + R_0^6 \rho V^2 (2S_{-1,2}^{3,-1} R_0 + S_{-2,3}^{5,-2} + R_0 S_{-2,3}^{6,-2})/2$, $\mathcal{C}_4 = \sigma R_0^2 S_{2,0}^{-1,0}/2$, with $S_{i,j}^{k,l} = R_0^{-l} \int_0^\pi h^l g^i g_\theta^j \sin^k\theta d\theta$. Terms proportional to $\eta \eta_\phi^2$ can be neglected since they introduce a factor η_0^3/L^2 which is small compared to η_0^3, i.e. it is in the third order. In order to verify the correctness of the above approximations, we present, for a typical soliton solution $\eta(\phi, t)$, some terms occuring in the expresion of E, Fig. 1. All details of calculation are given in [10,13]. Therefore, the energy of the non-linear liquid drop model can be interpreted as the Hamiltonian of the one-dimensional KdV equation. The coefficients in eq.(2) depend on two stationary functions of θ (the depth $h(\theta)$ and the transversal profile $g(\theta)$), hence, under the integration, they involve only a parametric dependence.

The KdV equation has the following cnoidal wave (Jacobi elliptic function) as exact solution

$$\eta = \alpha_3 + (\alpha_2 - \alpha_3) sn^2 \Big(\sqrt{\frac{C_3(\alpha_3 - \alpha_2)}{12 C_4}} (\phi - Vt) \Big| m \Big), \tag{3}$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants of integration, $m^2 = (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1)$. This solution oscillates between α_2 and α_3, with a period $T = 2K(m)\sqrt{\frac{(\alpha_3 - \alpha_2)C_3}{3C_4}}$, where $K(m)$ is the period of a Jacobi elliptic function $sn(x|m)$. The parameter V is the velocity of the cnoidal waves and $\alpha_1 + \alpha_2 + \alpha_3 = \frac{3(V-C_2)}{2C}$. In the limit $\alpha_1 = \alpha_2 = 0$ the solution eq.(3) approaches

$$\eta = \eta_0 sech^2\Big[\sqrt{\frac{\eta_0 C_3}{12 C_4}}(\phi - Vt)\Big], \tag{4}$$

which is the soliton solution of amplitude η_0. Small oscillation occur when $\alpha_3 \to \alpha_2$ and $m \to 0, T \to \pi/2$. Consequently, the system has two limiting solutions, a periodic and a localized traveling profile, which deform one into the other, by the initial conditions and the velocity parameter V. A figure showing the deformation from the $l = 5$ cnoidal mode towards a soliton is shown in Figs. 2.

The cnoidal solution eq.(3) depends on the parameters α_i subjected to the volume conservation and the periodicity condition of the solution (for the final soliton state this condition should be taken as a quasi-periodicity realised by the rapidly decreasing profile. This a problem of the basic model, [10]). The periodicity restriction reads

$$K\Big(\sqrt{\frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1}}\Big) = \frac{\pi}{n}\sqrt{\alpha_3 - \alpha_1}, \quad n = 1, 2, \ldots, 2\sqrt{\alpha_3 - \alpha_1}. \tag{5}$$

Hence, a single free parameter remains, which can be taken either one out of the three α's, V or η_0. Equatorial cross-sections of the drop are shown in Fig. 2b for the cnoidal solution at several values of the parameter η_0. All explicite calculations are presented in detail in [10].

In Fig. 3 we present the total energy plotted versus the parameters α_1, α_2 for constant volume. From the small oscillation limit ($\alpha_2 \simeq 3$ in the figure) towards the solitary wave limit ($\alpha_2 = 1$ in the figure) the energy increases and has a valley for $\alpha_1 \simeq 0.1$ and $\alpha_2 \in (1.2, 1.75)$ (close to the $l = 2$ mode). In order to introduce more realistic results, the total hydrodynamic energy is plotted versus α_1, α_2 for constant volume, too but we marked those special solutions fulfilling the periodicity condition. In Fig. 4 we present the total energy valley, from the small oscillations limit towards the solitary wave limit. We notice that the energy constantly increases but around $\alpha_2 \in (1.2, 1.75)$ (close to the linear $l = 2$ mode) it has a valley providing some stability for solitary solution (also called roton [13]).

3 The three-dimensional nonlinear model

In the following we introduce a sort of generalized KdV equation for fluids. We consider the three-dimensional irrotational flow of an ideal incompressible fluid layer in a semi-finte rectangular channel subjected to uniform vertical gravitation (g in z direction) and to surface pressure [12]. The depth of the layer, when the fluid is at rest is $z = h$. Boundary conditions at the finite spaced walls consist in annilation of the normal velocity component, i.e. on the bottom of the layer ($z = 0$) and on the walls $x = x_0 \pm L/2$ of the channel of width L. The following results remain valid if the walls expand arbitrary, e.g. $L \to \infty$, and the flow is free. We choose for the potential of the velocities the form

$$\Phi = \sum_{k \geq 0} \alpha_k(t) \cos \frac{k\pi(x - x_0)}{L} \cosh \frac{\sqrt{2}k\pi(y - y_0)}{L} \cos \frac{k\pi z}{L}, \tag{6}$$

where $\alpha_k(t)$ are arbitrary functions of time and L is a free parameter. Eq.(7) fulfils $\triangle\Phi = 0$ and the above boundary conditions at the walls. However there is another boundary condition at the free surface of the fluid [7]

$$(\Phi_z - \eta_t - \eta_x \Phi_x)_{z=h+\eta} = 0, \tag{7}$$

where $\eta(x, t)$ describes the shape of the free surface. By introducing the function

$$f(x, t) = \sum_{k=0}^{\infty} \frac{\alpha_k(t) k\pi}{L} \left(\sin \frac{k\pi(x - x_0)}{L} \cosh \frac{\sqrt{2}k\pi(y - y_0)}{L} \right), \tag{8}$$

the velocity field on the free surface can be written

$$\begin{aligned} \Phi_x|_{z=h+\eta} &= -\cosh(z\partial_x) f, \\ -\Phi_z|_{z=h+\eta} &= -\sinh(z\partial_x) f. \end{aligned} \tag{9}$$

Eqs (10) do not depend on L and the case $L \to \infty$ of unbounded channels and free travelling profiles remains equaly valid. Since the unique force field in the problem is potential, the dynamics is described by the Bernoulli equation, which, at the free surface, reads

$$\Phi_{xt} + \Phi_x \Phi_{xx} + \Phi_z \Phi_{xz} + g\eta_x + \frac{1}{\rho} P_x = 0. \tag{10}$$

Here P is the surface pressure obtained by equating P's first variation with the local mean curvature of the surface, under the restriction of the volume conservation

$$P\Big|_{z=h+\eta} = \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{3/2}}, \tag{11}$$

and σ is the surface pressure coefficient. The pressure in eq.(12) approaches $-\sigma\eta_{xx}$, for small enough relative amplitude of the deformation η/h. In order to solve the system of the two partial differential equations (8,11) with respect to the unknown functions $f(x,t)$ and $\eta(x,t)$, we consider the approximation of small perturbations of the surface compared to the depth, $a = max|\eta^{(k)}(x,t)| << h$, where $k = 0, ..., 3$ are orders of differentiation. Inspired by [12] and using a sort of perturbation technique in a/h, we obtain from eqs.(6-11) the generalised KdV equation

$$\eta_t + \frac{c_0}{h}\sin(h\partial)\eta + \frac{c_0}{h}(\eta_x \cosh(h\partial)\eta + \eta\cosh(h\partial)\eta_x)0. \tag{12}$$

If we approximate $\sin(h\partial) \simeq h\partial - \frac{1}{6}(h\partial)^3$, $\cosh(h\partial) \simeq 1 - \frac{1}{2}(h\partial)^2$, we obtain, from eq.(9), the polynomial differential equation:

$$a\tilde{\eta}_t + 2c_0\epsilon^2 h\tilde{\eta}\tilde{\eta}_x + c_0\epsilon h\tilde{\eta}_x - c_0\epsilon\frac{h^3}{6}\tilde{\eta}_{xxx} - \frac{c_0\epsilon^2 h^3}{2}\Big(\tilde{\eta}_x\tilde{\eta}_{xx} + \tilde{\eta}\tilde{\eta}_{xxx}\Big) \;=\; 0, \tag{13}$$

where $\epsilon = \frac{a}{h}$. The first four terms in eq.(20) correspond to the zero order approximation terms in f, obtained from the boundary condition at the free surface, i.e. the traditional way of obtaining the KdV equation in shallow channels.

In order to find an exact solution for eq.(12) we can write it in the form:

$$Ahu_X(X) + \frac{u(X+h) - u(X-h)}{2i} + u_X(X)\frac{u(X+h) + u(X-h)}{2}$$

$$+u(X)\frac{u_X(X+h) + u_X(X-h)}{2} = 0, \tag{14}$$

where $X = x + Ac_0t$ and A is an arbitrary real constant. We want to stress here that eq.(14) is a finite-difference differential equation, which is rather the exception than the rule fir such systems. Hence, it may contain among its symmetries, the scaling symmetry. Actualy, the first derivative of $u(X)$ is shown to be alinear combination of translated versions of the original function. In this way, the theory of such equations can be related with the wavelet, or other self-similarity systems, theory, [13]. In the following we study the solutions with a rapid decreasing at infinity and make a change of variable: $v = e^{BX}$ for $x \in (-\infty, 0)$ and $v = e^{-BX}$ for $x \in (0, \infty)$, with B an arbitrary constant. Writing $u(X) = -hA + f(v)$, and choosing for the solution the form of a power series in v:

$$f(v) = \sum_{n=0}^{\infty} a_n v^n, \tag{15}$$

we obtain a nonlinear recurrsion relation for the coefficients a_n:

$$\Big(Ahk + \frac{\sin(Bhk)}{B} \Big) a_k$$

$$= -\sum_{n=1}^{k-1} n \Big(\cosh\left(Bh(k-n)\right) + \cosh(Bh(k-1)) \Big) a_n a_{k-n}. \tag{16}$$

With the coefficients given in eq.(16) the general solution η can be written analyticaly. In order to verify the consistency of this solution we study a limiting case of the relation, by replacing sin and cosh expressions with their lowest nonvanishing terms in their power expansions Thus, eq.(16) reduces to

$$\alpha_k = \frac{6}{B^2 h^3 k(k^2-1)} \sum_{n=1}^{k-1} n \alpha_n \alpha_{k-n}, \tag{17}$$

and

$$\alpha_k = \left(\frac{1}{2B^2 h^3} \right)^{k-1} k \tag{18}$$

is the solution of the above recurrence relation. In this approximation, the solution of eq.(12) reads

$$\eta(X) \;=\; 2B^2 h^3 \sum_{k=1}^{\infty} k \Big(-e^{-B|X|} \Big)^k = \frac{B^2 h^3}{2} \frac{1}{\left(\cosh(BX/2)\right)^2}, \tag{19}$$

which is just the single-soliton solution of the KdV equation and it was indeed obtained by assuming h small in the recurrence relation (16). Hence, we have shown that the KdV equation describing the shallow liquids can be generalised for any depths and lengths. This result may be the starting point to search for more interesting symmetries. It would be interesting to interpret the generalized-KdV eq.(12) as the Casimir element of a certain algebra.

4 Compacton and self-similar solutions

Eq.(12) has a special character, namely contains both infinitesimal and finite difference operators. This particularity relates it to another field of nonlinear systems, that is scaling functions and wavelet basis, functions or distributions with compact support and self-similarity properties. In the following we investigate a particular case of eq.(12), that is when $h \ll \eta$, $h \ll \delta$, where δ is the half-width of the solution, if this has bounded or compact support. In this approximations, from eq.(12) we keep only the terms

$$\frac{1}{c_0}\eta_t + \eta_x + \frac{1}{h}\eta\eta_x - \frac{h}{2}\eta_x\eta_{xx} + \frac{1}{h}\eta\eta_x - \frac{h}{2}\eta\eta_{xxx} + \mathcal{O}_3 \simeq 0. \tag{20}$$

This equation is related to another intergable system, namely the K(2,2) equation, investigated in [11]

$$\eta_t + (\eta^2)_x + (\eta^2)_{xxx} = 0. \tag{21}$$

The main property of the K(2,2) equation is the equal occurence of non-linearity, dispersion and the existence of a Lagrangian and Hamiltonian system associated with it. The special solutions of this equation are the compactons

$$\eta_c = \frac{4\eta_0}{3}\cos^2\Big(\frac{x - \eta_0 t}{4}\Big), \quad |x - \eta_0 t| \geq 2\pi, \tag{22}$$

and $\eta_c = 0$ otherwise. This special solutions have compact support and special properties concerning the scattering between different such solutions. As the authors comment in [11], the robustness of these solutions makes it clear that a new mechanism is underlying this system. In this respect, we would like to add that, taking into account eq.(12), this new mechanism might be related to selfsimilarity and multiscale properties of nonlinear systems.

5 Conclusions

In the present paper we introduced a non-linear hydrodynamic model describing new modes of motion of the free surface of a liquid. The total energy of this nonlinear liquid drop model, subject to non-linear boundary conditions at the free surface and the inner surface of the fluid layer, gives the Hamiltonian of the Korteweg de Vries equation. We have studied the stability of the cnoidal wave and solitary wave solutions, from the point of view of minima of this Hamiltonian.

The non-linear terms yield rotating steady-state solutions that are cnoidal waves on the surface of the drop, covering continuously the range from small harmonic oscillations, to anharmonic oscillations, and up to solitary waves. The initial one-dimensional model [10] was extend to a three-dimensional model. A kind of new generalized KdV equation, together with some of its analytical solutions have been presented. We also found a connection between the obtained generalized KdV equation, and another one (i.e. K(2,2)), in a certain approximation. In this case, compacton solutions have been found and new symmetries (e.g. self-similarity) were put into evidence.

The analytic solutions of the non-linear model presented in this paper, make possible the study of clusterization as well as to explain or predict the existence of new strongly deformed shapes, or new patterns having compact support or finite wavelength. The model applies not only in fluid and rheology theories, but may provide insight into similar processes occurring in other fields and at other scales, such as the behavior of superdeformed nuclei, supernova, preformation of cluster in hydrodynamic models (metallic, molecular, nuclear), the fission of liquid drops (nuclear physics), inertial fusion, etc.

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FIGURE CAPTIONS

Fig. 1

The order of smallness of four typical terms depending on ϕ and occuring in the Hamiltonian, eqs.(1,2). Order zero holds for η^2, order 1 for η_ϕ^2, order 2 for η^3, and order 3 for $\eta\eta_\phi^2$.

Figs. 2

2a.

The transition of the cnoidal solution, from a $l = 5$ mode to the soliton limit: shape of the cross-section for $\theta = \pi/2$ function as a function of α_2 with $\alpha_{1,3}$ fixed by the volume conservation and periodicity conditions.

2b.

Cnoidal solutions (cross-sections of $\Sigma 1$ for $\theta = \pi/2$) subject to the volume conservation constraint. Results for the $l = 6$ mode to the $l = 2$ mode and a soliton are shown. The corresponding linear modes, i.e. spherical harmonics, are superimposed on the non-linear solutions.

2c.

Pictorial view of a soliton deformation of a drop, on the top of the original undeformed sphere. The supporting sphere for the soliton has smaller radius because of the volume conservation.

Fig. 3

The energy plotted versus α_1, α_2 for constant volume. From the small oscillation limit ($\alpha_2 \simeq 3$) towards the solitary wave limit ($\alpha_2 = 1$) the energy increases and has a valley for $\alpha_1 \simeq 0.1$ and $\alpha_2 \in (1.2, 1.75)$ (close to the $l = 2$ mode).

Fig. 4

The total energy plotted versus α_1, α_2 for constant volume (small circles). Larger circles indicate the patterns fulfilling the periodicity condition. From the small oscillations limit ($\alpha_2 \simeq 3$) towards the solitary wave limit ($\alpha_2 = 1$) the energy increases but for $\alpha_2 \in (1.2, 1.75)$ (close to $l = 2$ mode) it has a valley..