# Quasimolecular Resonances in Alpha+$^{20}$Ne Systems

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**Abstract:**

By introducing a Lagrangean depending of a complex scalar field a quantum analogue of the classical large amplitude collective motion in nuclei (solitonic excitations of the nuclear surface), successfully used for the evaluation of the preformation factors in alpha and cluster decays, was formulated. We have shown that in the classical limit the corresponding Lagrange equation, which is a modified nonlinear Schrödinger (MNLS) equation with solutions solitons and breathers, becomes a Korteweg de Vries (KdV) equation with only soliton solutions. By using a non perturbative weak-coupling procedure we have quantized the normal modes of the soliton and breather solutions. We have shown that, in the third order approximation, the corresponding Hamiltonian becomes diagonal with a spectrum similar with a sum of nonlinear harmonic oscillator spectra. In this way an additional degree of freedom and new quantum numbers are introduced. This formulation, applied for the description of the quasimolecular spectra explaines some of the observed levels and predicts also positions and spins of other levels, of both even and odd parities, with the parameters having the same values for all the levels.

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Short Title: **Quasimolecular resonances in $\alpha$+$^{20}$Ne Systems**

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## 1 Introduction

One of the interesting aspects of heavy ion nuclear physics is the existence of quasi-molecular structure. In such complex nuclear systems, there are groups of states with the structure polarized strongly into subunit nuclear clusters which can be defined as molecule-like structures [1]. Alpha-particle scattering has been known to be an important tool for studies of the general mechanism of reactions between complex nuclear particles and for exploring interesting nuclear structure properties. The alpha particle, being composite, interacts with nuclei in a more complex way than the nucleons. Its high stability (high binding energy, zero spin and isospin) restricts the interaction to a shallow surface layer region of the nucleus.

During the last decades, a big amount of data for elastic scattering of alpha particles on heavy or medium heavy nuclei has been published. These data showed rich resonant structure corresponding to many possible intermediate configurations. However some features are similar in all these experiments: a very high density of resonances (arround 25 observed states over 5 MeV range of the excitation energy), a good spin and parity assignment, the irregular spacing of these spectra and the relatively small moment of inertia of the $\alpha$+nucleus-systems. Hundreds of fragments of alpha cluster states have been reported in the elastic scattering of alpha particles on light nuclei over a broad region in excitation energy. These intermediate states of $\alpha$+nucleus could be also considered as belonging to an $\alpha$-molecular cluster structure. In the scattering from $^{40}$Ca [2], $^{28}$Si [3] and $^{20}Ne$ [4], for example, they occur as fragments of broad resonances forming groups with a common angular momentum in a mixed parity rotational band of excited states. Due to the widths and structure of the observed levels we consider that in such cases we have complicated alpha-cluster states rather than an interaction with the compound states.

There are many theoretical attempts (microscopical and phenomenological models) to explain such resonances or the intermediate structure (for several excellent review articles see the references in the Preface of [1]). Some of them were developed for $\alpha$-cluster states (rotational states fragmented by dipol [5] or radial vibrations [6] or by anharmonic quadrupole surface vibrations analogous to $\beta$ and $\gamma$ vibrations in deformed nuclei [7], Morse-potential model [6], the quadrupole vibration-rotation model [7, 8], coupled-channel calculations [9], local-potential models, etc.). Different models were developed for other quasimolecular resonances: the two-center shell model [10], the vibron model and semimicroscopic algebraic models [5, 11], molecular models of two fragments capable of collective motions [12], the double-resonance model (see [1]), band-crossing models [13], RGM and OCM model [14 and the references herein], GCM model [15, and the references herein], etc., and many other possible explanations [16-22]. The theoretical energy spectra obtained in these models were successfully used for the description of positive parity resonant states for two colliding carbon nuclei and for bound alpha cluster states in $\alpha+^{12}$C and $\alpha+^{16}O$ [23]. The great number of states as well as their irregular appearance, and the relatively small moment of inertia of the $\alpha$+nucleus-systems [3] call for other models.

On another hand it is known that the shell model can not describe the cluster aspects of nuclei [24]. Microscopically, in order to explain such effects, one has to introduce in the shell model a cluster-like component [25] or the many-body correlations. We consider that a collective description, in which the outside nucleons of a rigid core join together to form a cluster, could also represent an appropriate frame for the evaluation of cluster preformation factors [26].

In this paper (and in two other recent ones, [27,28]) we shall address the problem of obtaining an expression for the energy spectra of such $\alpha$+cluster resonances. We introduce here another possible mode of explanation of these spectra, in the frame of a certain Lagrangean model. We use a nonlinear Lagrangean for the description of the system in terms of the dynamics of the nuclear surface, in the background of the liquid drop model. This dynamical description of these states was first introduced [26,27] by considering that in the intermediate states the alpha particle interacts with the nucleus as a soliton or a breather. In this description we assume that the outside nucleons do not polarize the core and these nucleons join together due to the nonlinear effects, i.e. the stability of the solitonic shapes. Consequently in this collective description new shapes appear which are hardly described by a simple multipole expansion of the nuclear surface. E.g. for the description of a solitonic shape on a surface one needs at least $l = 10$ spherical harmonics, for a resonable range of errors [26, last reference]. Recently [26], we have shown that by considering the nonlinear terms in the classical hydrodynamic equations, stable solitons exist on a circle with the shape $r = R + \eta(\theta, t)$ respectively on a sphere if the perturbation has an axial symmetry. The soliton as a perturbation of the shape $\eta(\phi, t)$, is described by the Korteweg-de Vries (KdV) equation, and the breather (modulated soliton) is described by the Nonlinear Schrödinger equation (NLS) [29]. The conserved energy (the corresponding Hamiltonian of the nonlinear system), under the condition of constancy of the volume of the soliton-like alpha particle, and the phenomenological shell corrections for the core and the soliton, lead to a new minimum on the total energy surface as a function of the new collective coordinates. Due to the fact that alpha and cluster decays are spontaneous decays we have chosen the minimum to be degenerated with the ground state minimum. The penetrability between the two minima give the probability of the formation of the soliton on the nuclear surface. In this way we introduced a new coexistence model consisting of the usual shell model and a cluster-like model describing a soliton moving on the nuclear surface. This model already was shown to describe excellently the experimental preformation factors for alpha and cluster [26] decays. A similar model was introduced microscopically in ref. [25] and, early, by Könnecke et. al. [30].

Following the above line, in the present paper we introduce a quantum nonlinear analogue of the above classical hydrodynamic description of the soliton with the help of a Lagrangean depending of a complex scalar field. The corresponding Lagrange equation is a modified nonlinear Schrödinger (MNLS) equation. The resulting Hamilton equations lead to the same MNLS equation. We have shown that by assuming a traveling wave profile this equation becomes a MKdV equation which, by a nonlinear Miura transformation, yields finally in a KdV equation. Consequently, a full correspondence with the previous classical description is obtained [26]. The quantum fluctuations around the clasical solutions (i.e. solitons and breathers) are obtained by a non-perturbative weak-coupling procedure, which consists in a functional Taylor expansion of the potential up to third order around the static solutions. A parallel between the MNLS and KdV equations is shown to exist at the classical and quantum level. We diagonalize the Hamiltonian, and the obtained spectra are similar with a sum of nonlinear harmonic oscillators spectra determined uniquely by the soliton geometry.

The above quantum approach of cluster formation on the nuclear surface was successfully applied to the description of the resonances observed in the elastic scattering of alpha particles on $^{28}Si$ [28]. In the present approach we find a generalisation of this previous model and we apply also this description to the resonances observed in the elastic scattering of alpha particles on $^{20}Ne$. We assume that the free alpha particle, in contact with the nucleus, looses its entity and becomes a breather state with the preformation factor 1. At present this model can reproduce and predict the energies and the spins of the resonances, but can not give widths, because the resonance states are treated as bound states.

We concluded that the quasimolecular spectra can be described in the present model by a rotational part $\frac{\hbar^2}{2\mathcal{I}}I(I+1)$ and a solitonic part $\sum_k \hbar\omega_k\left(N_k+\frac{1}{2}\right)+\sum_k B_k\left(N_k+\frac{1}{2}\right)^2$ which contains a finite number of nonlinear harmonic oscillators with new quantum numbers $N_k$, energies $\hbar\omega_k$ and quadratic corrections $B_k$ determined by the soliton geometry and no other free parameters are introduced. Evidently such a possible description is limited only to the quasimolecular spectra observed in the collision between two asymmetric nuclei.

## 2 Solitons and breathers on the nuclear surface

Solitons and breathers are special solutions (solitary waves) of nonlinear partial differential equations like Korteweg-de Vries (KdV), Modified Korteweg-de Vries (MKdV), Nonlinear Schrödinger (NLS) equations [29]. The common feature of these nonlinear equations is the fact that they describe dynamical systems, i.e. their general form is $u_t(x,t)=\mathcal{O}u(x,t)$ where the left-hand side represents the derivative of the unknown function $u$ with respect to time and $\mathcal{O}$ is a nonlinear differential operator with respect to the spatial coordinate, acting on the same function. These solitary waves are non-dispersive localised packets of energy moving uniformly and resembling extended particles. The properties of these equations are well known: both nonlinear and dispersive. Consequently, due to the balance of these effects, some of their solutions travel in space without distorsion in shape, i.e. their localized solutions are stable. The condition of the localization of the energy density asks for the introduction of the Lagrange formalism. In the following we apply such a formalism to the classical nonlinear large amplitude collective motion which was recently introduced [26] for the description of the formation of the clusters on the nuclear surface. We introduce a class of Lagrangeans with the property that the solutions of their Lagrange equations are reducible to the KdV or NLS equations. Consequently we can make the approach towards the nonlinear hydrodynamic model.

The first step is to consider a model physical system described by a complex field $u(\phi,t)$ defined on a circle of radius $R$, $\phi\in[0,2\pi)$. This is an infinite-dimensional Hamiltonian system characterised by a Lagrangean depending on the field $u(\phi,t)$ and its complex conjugate, in the general form:

$$L[\phi,\phi^*] = \int_0^{2\pi} \mathcal{L}(u,u^*)d\phi = \int_0^{2\pi} \left[\frac{1}{\lambda}|u_t|^2 - \frac{i\mu}{2}(uu_t^* - u^*u_t) + |u_\phi|^2 - \frac{1}{4}|u|^4 + 4\beta^2|u|^2\right]d\phi \quad (1)$$

where $\phi$ represents the angular coordinate on the circle, $\lambda = c^2/R^2$, $c$ is the velocity of light, $\beta$ is a real parameter similar with the mass parameter in $\Phi^4$ field theory [31,32] and $\mu$ another free parameter. The corresponding Lagrange density equation for the Lagrangean density $\mathcal{L}$ is

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t^*}\right) + \frac{\partial}{\partial \phi}\left(\frac{\partial \mathcal{L}}{\partial u_\phi^*}\right) - \frac{\partial \mathcal{L}}{\partial u^*} = 0, \quad (2)$$

which leads to the following modified nonlinear Schrödinger equation (MNLS)

$$\frac{1}{\lambda}u_{tt} - i\mu u_t = -u_{\phi\phi} - \frac{1}{2}|u|^2u + 4\beta^2 u. \quad (3)$$

This equation is a generalization of the NLS equation in the sens that contains a first derivative (like in the case of NLS equation or any other dynamical equation) and a second derivative (like in the case of wave equation) with respect to time. In the limit $\lambda \to \infty$, $\mu = -1$ and $\beta = 0$ we reobtain the NLS equation

$$iu_t = -u_{\phi\phi} - |u|^2 u. \quad (4)$$

and in the limit $\mu = 0$ we obtain again a nonlinear wave equation. Evidently a similar equation is valid for $u^*(\phi, t)$ (the complex conjugated form). From the Lagrangean density equation (1) we obtain the corresponding energy density:

$$\begin{aligned}\mathcal{H}(u,u^*) &= u_t\frac{\partial \mathcal{L}}{\partial u_t} + u_t^*\frac{\partial \mathcal{L}}{\partial u_t^*} - \mathcal{L} \quad (5)\\ &= \frac{1}{\lambda}|u_t|^2 - |u_\phi|^2 + \frac{1}{4}|u|^4 - 4\beta^2|u|^2,\end{aligned}$$

with the property that its integral, the Hamiltonian, is conserved as $t$ varies, and gives the total energy functional $E$

$$E[u,u^*] = \int_0^{2\pi} \mathcal{H} d\phi = T + U. \quad (6)$$

Here we defined the total kinetic energy $T$ and the total potential energy $U$ from $L = T - U$ and eq.(5). The Lagrangean can now be written as a sum of three terms

$$L = T + \mu Q - U \quad (7)$$

with

$$T[u,u^*] = \frac{1}{\lambda}\int_0^{2\pi} |u_t|^2 d\phi, \quad (8)$$

$$U[u,u^*] = \int_0^{2\pi}\Big(-|u_\phi|^2 + \frac{1}{4}|u|^4 - 4\beta^2|u|^2\Big)d\phi. \tag{9}$$

The remaining term in eq.(7)

$$Q(u,u^*) = -\frac{i}{2}\int_0^{2\pi}(uu_t^* - u^*u_t)d\phi, \tag{10}$$

corresponds to another conserved quantity, the conserved topological charge $Q$ [32]. The conservation of $Q$ is evident from the $U(1)$ global symmetry of the Lagrangean (1). Being a conserved quantity $Q$ does not appear in the expression of $H$. By introducing the conjugate momenta

$$u_t = \lambda\Big(p^* + \frac{i\mu}{2}u\Big) \quad ; \quad u_t^* = \lambda\Big(p + \frac{i\mu}{2}u^*\Big), \tag{11}$$

the kinetic energy becomes

$$T = \lambda\int\Big(pp^* - \frac{i\mu}{2}p^*u + \frac{i\mu}{2}pu - \frac{1}{4}uu^*\Big)d\phi. \tag{12}$$

Consequently the Hamilton equations

$$\begin{aligned} u_t = \frac{\partial\mathcal{H}}{\partial p} \quad &; \quad u_t^* = \frac{\partial\mathcal{H}}{\partial p^*} \\ p_t = -\frac{\partial\mathcal{H}}{\partial u} \quad &; \quad p_t^* = -\frac{\partial\mathcal{H}}{\partial u^*} \end{aligned} \tag{13}$$

give directly the same MNLS eq.(3)

$$\frac{1}{\lambda}u_{tt} - i\mu u_t = \frac{\partial\mathcal{U}}{\partial u^*}. \tag{14}$$

In the following we show how the solutions of the MNLS eq.(3) are connected with solitons and breathers. We find for eq.(3) two classes of localized solutions: quasi-static solutions and traveling wave solutions. We define the quasi-static solutions like:

$$u(\phi,t) = u_0(\phi)e^{i\alpha t}, \tag{15}$$

with $\alpha$ a parameter which labels these solutions. By introducing the solution (15) in eq.(3) we obtain the following differential equation for $u_0(\phi)$

$$u_{0\phi\phi} = \Big(-\alpha\mu + 4\beta^2 + \frac{\alpha^2}{\lambda}\Big)u_0 - u_0^3, \tag{16}$$

with the general solution in the form

$$u_0(\phi) = \pm\sqrt{2\Big(-\alpha\mu + 4\beta^2 + \frac{\alpha^2}{\lambda}\Big)}\,sech\Big(\sqrt{-\alpha\mu + 4\beta^2 + \frac{\alpha^2}{\lambda}}\,\phi\Big). \tag{17}$$

With the condition $\alpha\mu - 4\beta^2 - \frac{\alpha^2}{\lambda} > 0$ this solution yields a static soliton for $|u|^2$ and a breather-like solution for $Re(u)$ [29]

$$|u|^2 = 2\Big(-\alpha\mu + 4\beta^2 + \frac{\alpha^2}{\lambda}\Big) sech^2\Big(\sqrt{-\alpha\mu + 4\beta^2 + \frac{\alpha^2}{\lambda}}\ \phi\Big), \tag{18}$$

$$Re(u) = \sqrt{2\Big(-\alpha\mu + 4\beta^2 + \frac{\alpha^2}{\lambda}\Big)} sech\Big(\sqrt{-\alpha\mu + 4\beta^2 + \frac{\alpha^2}{\lambda}}\ \phi\Big) cos(\alpha t). \tag{19}$$

We note that LHS of eq.(3), i.e. the time depending part, cancels for $\alpha = \lambda\mu$. This case corresponds to the static solution of MNLS and will be used further in the quantizing procedure.

On the other hand, we can consider a more general solution of eq.(3) in the form of a traveling wave packet with half width $L$ and the speed $V$

$$u(\phi, t) = r\Big(\frac{\phi - Vt}{L}\Big) e^{i(a\phi + bt)}, \tag{20}$$

where $r$ is an analytical function and $a$ and $b$ are constants. In this case, by taking $\mu = 1$, and by introducing the explicite dependence of $\lambda$ as a function of $R$, we obtain, from the real part of eq.(3), by differentiation with respect to $\phi$, the MKdV equation for $r$:

$$\frac{4\beta^2 + a^2 + Rb/c - R^2b^2/c^2}{V} r_t + \frac{3}{2} r^2 r_\phi + \Big(1 - \frac{R^2V^2}{c^2}\Big) r_{\phi\phi\phi} = 0, \tag{21}$$

with the solution

$$r = r_0 sech\Big(\sqrt{\frac{4\beta^2 + a^2 + Rb/c - R^2b^2/c^2}{1 - R^2V^2/c^2}} (\phi - Vt)\Big), \tag{22}$$

and its amplitude

$$r_0 = 2\sqrt{-R^2b^2/c^2 + a^2 + b + 4\beta^2}. \tag{23}$$

By introducing eqs.(22,23) in eq.(21) and by using now the imaginary part of eq.(3)

$$(1 - 2bR^2/c^2) r_t = -2a r_\phi,$$

we determine the parameters $V$, $a$ and $b$ as functions of $r_0$, $L$ and $\beta$ only:

$$V = c\sqrt{4 - L^2 r_0^2}/2R,$$

$$a = \sqrt{(r_0^2 - 16\beta^2)(L^2 r_0^2 - 4)}/2Lr_0,$$

$$b = c\sqrt{r_0^2 - 16\beta^2}/RLr_0.$$

By analysing these last expressions for the velocity of the shape $V$ and the parameters $a$ and $b$ we find that all the traveling wave solutions of eq.(3) are classified into four types, depending on the values of the parameters $x = r_0/4\beta$ and $y = Lr_0/2$. The parameter $x$ controls the stability of the solution against decaying to zero and $y$ controls the modulation in time and space of these solutions. An example is presented in Fig.1, where typical solutions $u(\phi, t)$ given by eq.(21) are ploted at three different moments of time. For simplification the coordinate $\phi$ was drawn as a line. The horizontal axis represents the time evolution. For $x > 1$ we have instable solutions which can decay into small normal modes of oscillation (linear waves), i.e. for $y < 1$ (Fig.1a) a soliton-like solution and for $y > 1$ (Fig.1b) a breather-like solution. For $x < 1$ we have stable solutions in the form of a traveling oscillating soliton $y > 1$ (Fig.1c) and breather solutions $y < 1$ (Fig.1d). Consequently, the range of physical interest of the parameters is given by the stability condition for the solitary waves $r_0 \leq 4\beta$. We stress that by the continuous variation of the parameter $x$ we can describe in this model both nonlinear solutions of the dynamics of the surface and the linear dynamics of the traditional liquid drop model as an asymptotical limit of this model. In order to obtain this limit we have to reduce the amplitude $r_0$ of the solution such that the last two terms of the RHS of eq.(3) could be treated as small terms. Consequently, if we choose $\beta = 0$, $\mu = 0$ and $r_0 \to 0$ we obtain the conditions $L = 2$ and $a = \sqrt{r_0^2 - 1}/2$ and $b = c/2R$. In this case eq.(3) reduces to the wave equation on the circle with the corresponding normal modes as solutions.

In order to make the connection with the classical hydrodynamic approach [26,27] we apply the complex nonlinear Miura transformation [29] to the solution given by eq.(20)

$$\eta(\phi, t) = u(\phi, t)^2 \mp i u_\phi(\phi, t). \tag{24}$$

In this case it is known that $\eta$ is a solution of the KdV equation which describes the classical soliton. The parameter $r_0$ can not be correlated with the soliton amplitude due to the fact that the amplitude of the field $u$ can not be interpreted as the probability of localization as eq.(3) is not a Schrödinger equation (does not describe a Fermi field). We note only that in the time independent limit ($V = a = b = 0$) we obtain a simple connection between the amplitude of the soliton and the parameter $\beta$, $4\beta = r_0$. In this case the remaining parameters $r_0$ and $L$ are determined only by the geometry of the corresponding soliton which overlaps the alpha volume. No other free parameters are left in this approach. If we investigate the time dependent solutions the parameter $\beta$ could be fixed by a similar condition, i.e., by comparing the shapes of such solutions with the geometry of the soliton overlaping an alpha particle. Finally we note the role of the topological charge invariant term $Q$ in the Lagrangean. If we choose $\mu = 0$ in eq.(3) than the parameters $a$ and $b$ become zero and the oscillating behaviour of the solution desappears. The corresponding solutions are still time dependent and represent a soliton (a breather) moving on the nuclear surface and having a constant envelope.

## 3 Quantization of the soliton and breather solutions

In the previous Section we investigated some nonlinear classical solutions of the hydrodynamical equations in terms of a nonlinear Lagrangean. At the quantum level the differences between linear and nonlinear behaviour of the system become larger. The basic elements of the linear approach (superpositions of states, wave packets, stationary states) are replaced in the nonlinear quantum theories with other formulations (spaces of solutions which are no more vector spaces, topological criteria of clasification of the states, i.e. sectors, stability criteria, etc.). In particular, several systems may be simple described in terms of an appropriately quantized state of a type of classical solutions to a field theory known as an "quantum extended objects" theory [33]. In the following we define such a quantum extended object as a classical solution of the Lagrange equations (i.e. of the dynamical equations of a local field theory) with localized energy density in a certain region of space. In the case of the solitons or breathers on the nuclear surface this definition implies localized shapes with the angular half-width considerable smaller than $2\pi$. The space distribution of the energy density is obtained by introducing these solutions in the corresponding functional Hamiltonian. In order to quantize such systems it was shown [32] that by using semiclassical expansion methods of quantum field theory one can associate quantum extended particles to the classical solutions together with their excited states. These quantum states are non-perturbative. Consequently the corresponding solutions (solitons, breathers, instantons) are characterized by new quantum numbers without analogue in linear quantum theory (e.g. topological index). This nonlinear quantum mechanical approach (the quantum theory based on solitons and instantons) does not change the main results of the linear theory (being non-perturbative), but results in a generalisation of its familiar ideas to field theory. We note that, in general, such nonlinear quantum mechanics approach gives results if some non-trivial classical solutions of the system are known [32].

In order to quantize the classical soliton and breather solutions of the nonlinear Lagrange equation we use the well known non perturbative weak-coupling procedure which consists in finding the normal modes of oscillation about the lowest energy state of the classical soliton or breather, exactly quantizing these normal modes, and finally doing perturbation theory in the nonlinear couplings among the normal modes. We begin by quantizing the Lagrangean (1) in the condition $\mu = 0$. We impose this condition because when we finally express the Hamiltonian in terms of the normal modes the topological charge term introduces a strong coupling even in the second order. Consequently we can not find a simple way of quantizing this Hamiltonian as a sum of harmonic oscillators. However, this term breaks the Galilean invariance of the remaining Lagrangean and consequently introduce a mixing between the fermionic and bosonic character of the Lagrange equation. We write the potential energy in eq.(9) as a functional Taylor expansion about the solutions which are its extrema. These are given by the equation:

$$\frac{\delta \mathcal{U}(u,u^*)}{\delta u^*} = u_{\phi\phi} + \frac{1}{2}|u|^2 u - 4\beta^2 u = 0. \tag{25}$$

The above equation has two classes of solutions. Trivial static solutions

$$u_{0k}^{vac}(\phi) = \sqrt{2(k^2+4\beta^2)}e^{\pm i(k\phi+\phi_{0k})}, \tag{26}$$

depending on the integer label $k$ and on an arbitrary phase factor $\phi_{0k}$, and non trivial ones

$$u_0(\phi,t) = 4\beta sech(2\beta\phi). \tag{27}$$

connected with the solitons. The functional Taylor expansion $u(\phi,t) = u_0(\phi,t) + \xi(\phi,t)$ for the potential $\mathcal{U}(u,u^*)$ in second order in $\xi$ gives

$$\mathcal{U}(u,u^*) = \mathcal{U}(u_0,u_0^*) + \mathcal{U}_2(\xi,\xi^*) + corrections. \tag{28}$$

Evidently, the first term $\mathcal{U}_1(\xi,\xi^*)$ is zero due to the extremum conditions for the potential. The second order term $\mathcal{U}_2(\xi,\xi^*)$ can be written in the integral form

$$U_2(\xi,\xi^*) = \int \xi^*\Omega\xi d\phi + c.c. \tag{29}$$

where $\Omega$ is a hermitian operator

$$\Omega = \frac{1}{2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2}|u_0|^2 + \frac{1}{4}u_0^2\mathcal{C} - 2\beta^2 \tag{30}$$

in which $\mathcal{C}$ denotes the complex conjugation (c.c.) operator, i.e. $\mathcal{C}\xi = \xi^*$. In order to find the fluctuations around the solutions $u_0$ and $u_0^*$ we have to find the eigenvalues $\omega_k^2 > 0$ and eigenfunctions $\xi_k$ of the operator $\Omega$, $\Omega\xi_k = \omega_k^2\xi_k$. We consider only positive eigenvalues such that the static solution must be stable against the fluctuations. Due to the fact that $\Omega$ is hermitian, its eigenfunctions form a complet orthonormal set of "normal modes" of fluctuations around the static solution. The most general perturbation can be expressed then like

$$\xi(\phi, t) = \sum_k C_k(t)\xi_k(\phi, t). \tag{31}$$

where $C_k(t)$ are Hermitian quantum operators. With this choice for $\xi$, the Hamiltonian (6) assumes a simple form in the second order:

$$\begin{aligned} H(u_0 + \xi, c.c.) &= T(u_0 + \xi, c.c.) + U(u_0 + \xi, c.c.) \\ \simeq H^2(u_0, \xi) &= U(u_0, u_0^*) + \sum_k \left( \left| \frac{dC_k}{dt} \right|^2 + \omega_k^2 \frac{c^2}{R^2}|C_k|^2 \right), \end{aligned} \tag{32}$$

which reduces the problem to a set of independent harmonic oscillators with the energy spectrum given by:

$$E^{0,2} = U(u_0, u_0^*) + \sum_k \hbar\tilde{\omega_k}\left(N_k + \frac{1}{2}\right) \tag{33}$$

where $\tilde{\omega_k} = c\omega_k/R$ and the $N_k$'s are new quantum numbers associated with the excited states of the vacuum of the system or of the soliton. The range of $k$ is fixed by the condition $\omega_k^2 > 0$.

If we take into account the higher orders, i.e. the third and the fourth order in $\xi$, $\xi^*$, we obtain, following the same algorithm of functional derivatives, the next higher order terms in the form

$$U^{3,4} = \frac{1}{2}\int_0^{2\pi}(u_0^*\xi^*\xi^2)d\phi + \frac{1}{8}\int_0^{2\pi}|\xi|^4 d\phi + c.c. \tag{34}$$

The contribution of these terms is no more diagonal and mixes the different states, with different $k$-values. Consequently, we can treat these higher orders only in the frame of stationary perturbation theory. We can write the final form of the Hamiltonian, up to the fourth order, from eqs.(32) and (34) as a sum between a non-perturbative diagonal term and a perturbation

$$H^4 \simeq H^2 + U^{3,4} \tag{35}$$

In the following we first investigate the eigenvalue problem for $H^2$. The trivial solutions (26) describe the vacuum associated with this theory and its excitations. However the vacuum sector of these energies and solutions is unconnected with the sector of nontrivial solutions because its states are orthogonal to the states of the soliton or breather solutions and, furthermore, not connected by any localised operator to them. In particular, the quantum soliton is stable against decay into mesons (the states can not transform one into another, except for the action of the instanton operator). The static solutions given by eq.(27) form an infinite set of discrete functions and for each such solution we have an unique eigenvalue problem $\Omega\ \xi_k^{vac} = \omega_{0k} u_{0k}^{vac}$. The corresponding eigenvalues have the form $\omega_{0k}^2 = k^2 + 2\beta^2$. We can now construct a "tower" of approximate harmonic oscillator states arround each $u_{0k}^{vac}$ with the spectrum given by

$$E_{vac} = \sum_k \hbar \tilde{\omega}_{0k}^2 \left( N_k + \frac{1}{2} \right), \tag{36}$$

where the tilda symbol has the same semnification as in the previous equations. We can interpret these quanta as the mesons of the vacuum of the system due to the fact that the excitation energy of the $k^{th}$ mode is $\hbar\omega_{0k} = \hbar\sqrt{(k^2c^2/R^2 + 2c^2\beta^2/R^2)}$ which is similar with the kinetik energy of a meson with momentum $\hbar ck/R$ and the mass $\sqrt{2}\beta/R$. A possible interpretation for the parameter $\beta$ is supported now by this last observation.

In the following we investigate the nontrivial solution $u_0$, eq.(27). In order to solve the eigenvalue problem we look for $\xi_k$ in the form

$$\xi_k(\phi) = sech^{\gamma_k}(2\beta\phi) F(2\beta\phi), \tag{37}$$

where $\gamma_k$ is a parameter and $F$ an arbitrary function. This leads to the following eigenvalue equation for $\Omega$

$$\frac{1}{2}\xi_k^{''} + \frac{3}{4}u_0^2\xi_k = (\omega_k^2 + 2\beta^2)\xi_k. \tag{38}$$

By using the substitution $v = \frac{1}{2}[1 + th(2\beta\phi)]$ we transform eq.(38) in a hypergeometric differential equation for $F$

$$v(v-1)F^{''} + \left[ \left( 2 + \frac{\gamma_k}{\beta} \right) v - \left( 1 + \frac{\gamma_k}{2\beta} \right) \right] F^{'} + \frac{\gamma_k(\gamma_k + 1) - 6}{4\beta^2} F = 0 \tag{39}$$

with the eigenvalues given by $\omega_k^2 = 2\beta^2(\gamma_k^2 - 1)$. In order to have localized solutions (rapidly decreasing functions for large $v$), the asymptotic behavior of the hypergeometric function $F$ [34] asks for the condition:

$$\gamma_k = -\frac{1}{2} - 2k\beta \pm \sqrt{\frac{25}{4} - 4k\beta^2 + 2k\beta} \tag{40}$$

where $k = 0, 1, ..$ such that $\omega_k^2 > 0$. The remaining spectrum of $\Omega$ is continuous. The spectrum associated with the excitations of the soliton is given by the expression eq.(33), the value of the $\omega_k$ given above.

We stress that the eigenvalues $\omega_k$ are uniquely determined by $\beta$. The states of the theory, $\Psi^{sol}_{N_k,k}(\phi, t)$, are defined as eigenstates of the Hamiltonian associated with expression (5) and by the occupation numbers $N_k$ of the different modes $k$. The wave functions associated with the excitations of the soliton have the form:

$$u_k(\phi) = u_0(\phi) + \sum_k sech^{\gamma_k}(2\beta\phi)F(a_k, b_k c_k v), \tag{41}$$

where $F$ is the hypergeometric function in two arguments: $a_k = -k$, $b_k = (6 - \gamma_k(\gamma_k + 1))/4\beta^2 k$, $c_k = 1 + \gamma_k/2\beta$. By taking into account the quasi-periodicity condition for the solitonic solutions in eqs.(27,37), the halh-width $L$ must lay in the range $(0, \pi)$ and consequently the parameter $\beta$ is limited to the range $(1/2\pi, \infty)$. The stability condition for the perturbed solutions ($\omega_k^2 > 0$) drastically limits this domain to $\beta \in (0.15, 0.9)$. The solutions with the negative sign are eliminated from these reasons. As an example we present in Fig.2, for the different values of the occupation numbers $k$, the corresponding solitonic wave functions, eq.(41), ploted on the circle. One can see how the solitonic solution (k=0) is modulated for higher occupation numbers ($k = 1, 2$) into a breather solution keeping the same solitonic envelope.

The calculation of the correction terms, given by the third and fourth orders follows from a perturbation technique applied to the Hamiltonian (32) with the perturbation given by eqs.(34), (35). We investigate here only a special case, i.e. of the geometry imposed by the $\alpha$+$^{20}$Ne system. An example of such a calculation is given in the next section. In this case one can approximate eq.(34) with a third order term in $|\xi|$ and consequently we can apply in eq.(35) the usual perturbation technique for third order perturbations of an harmonic oscillator, [7,35].

In the last part of this Section we like to present a similar Lagrangean approach for the Korteweg de Vries (KdV) equation. We introduce, instead of eq.(1), the Lagrangean density

$$\mathcal{L} = \frac{1}{2}u_\phi u_t - \frac{1}{2}u_\phi^2 + u_\phi^3 - u_{\phi\phi}^2. \tag{42}$$

After the substitution $\eta(\phi, t) = u(\phi, t)_\phi$ we obtain the corresponding Lagrange equation in the form

$$\eta_t - \eta_\phi + 6\eta\eta_\phi + \eta_{\phi\phi\phi} = 0, \tag{43}$$

which is exactly the KdV equation. Consequently one possible stationary solution of eq.(43) is a soliton

$$u_0(\phi) = \frac{1}{2} sech^2(\frac{\phi}{2}). \tag{44}$$

Following the same above presented formalism we expand the potential energy around this stationary solution and find for the operator $\Omega$ in this case the expression

$$\Omega = \frac{1}{2}\frac{\partial^4}{\partial_\phi^4} + 3u_{0,\phi}\frac{\partial^2}{\partial_\phi^2} + \Big(3u_{0,\phi\phi} + \frac{1}{2}\Big). \tag{45}$$

By solving the eigenvalue problem for this operator we obtain the orthonormal basis for the perturbation around the soliton solution, exactly like in the previous case for the MNLS equation. With a suitable choise for the coefficients of the different terms in the Lagrangean (42) we obtain a similar spectrum with that given by eq.(33). We want to stress that, though we have introduced here another type of Lagrangean, the model still is tractable and gives similar results with the first one. What is essential in both these approaches is the introduction of the nonlinear terms and the existence of the breather and/or solitonic stationary solutions.

## 4 Quasimolecular energy spectra for the $\alpha$+$^{20}$Ne system

In the following we apply the above quantum nonlinear approach of clusters in nuclei for the description of quasimolecular spectra observed in elastic scattering of two colliding nuclei. We would like to stress again that basically in our quantum nonlinear approach we assume that alpha particles in contact with the nuclei loose their entity and become breather states (i.e. bounded states). If this hypothesis is valid we have to use for the expression of the energy the quantum spectrum (33), i.e. the new quantum numbers $N_k$, in the second order of approximation.

Recently we have shown that such spectra are in very good agreement with experimental data for the scattering of alpha particles on $^{28}$Si [28]. In that case we had to add to the solitonic spectrum two other terms, i.e. another harmonic oscillator term and a quadratic perturbation. These terms were believed to be obtained from the Lagrangean theory of the soliton.

In the present approach we can show that we obtain the complete structure of this spectrum by using only the energies obtained from the quantization of the soliton. In this respect we take here into account the higher corrections in the energy, up to the third order. Due to the fact that the absolute value of the coefficients $\gamma_k$ increase with the increasing of $k$, eq.(40), and the support of the hypergeometric functions in eq.(41) decreases strongly with $k$ eqs.(39,40), [34], we make an approximation in eq.(35): we use only the first term in the RHS of eq.(35), i.e. the third order correction, and we approximate this term with: $\sum_k const.|C_k|^3$ . This approximation is reasonable because we take into account that the wave functions (41) strongly decrease in modulus with the increasing of $k$. This allows us to consider in the third and fourth order corrrections in eq.(35) only the highest terms, i.e. the diagonal ones. Consequently, we can write again, in this approximation, the Hamiltonian in the form:

$$H^3 \simeq U(u_0, u_0^*) + \sum_k \Big( \Big| \frac{dC_k}{dt} \Big|^2 + \omega_k^2 \frac{c^2}{R^2} |C_k|^2 \Big) + \frac{1}{2} \sum_k B_k \hbar \omega_k \Big( |C_k|^3 \Big). \tag{46}$$

This Hamiltonian represents a sum of independent anharmonic oscillators. Its corresponding spectrum is given, [35], in the first-order perturbation calculus, by

$$E^3 \simeq U(u_0, u_0^*) + \sum_k \hbar \tilde{\omega_k} \Big( N_k + \frac{1}{2} \Big) - \sum_k \Big[ \frac{15}{4} B_k^2 \hbar \omega_k \Big( N_k + \frac{1}{2} \Big)^2 + \frac{7}{16} \hbar \omega_k B_k^2 \Big]. \tag{47}$$

For the elastic scattering of alpha particles on $^{20}Ne$ the total wave function for the $\alpha +^{20} Ne$-system, taking into account the global rotation of the system, is

$$\Psi_{I,N_k}(\tilde{\theta}, \tilde{\phi}, \phi, r, t) = \Psi_{rot,I}(\tilde{\theta}, \tilde{\phi}, t) \Psi_{sol,N_k}(r, \phi, t) \tag{48}$$

The polar coordinates denoted with tilde are those associated with the global rotation.

Like in the previous work [28] the $\alpha$-particle in its cluster states is a soliton with a height $r_0 R_{^{20}Ne}$ and an angular half-width $L$. According to the dynamics of the soliton the $\alpha$-cluster moves on a shallow liquid layer arround a rigid spherical core, with a depth $h < R_{^{20}Ne}$. The parameters $r_0$ and $L$ are fixed, for each geometry, by imposing the same two conditions as in the earlier work [28]: the mass of the soliton is equal with the mass of the $\alpha$-particle ($A_{sol} = 4$) and we ask for the maximum overlap between the soliton shape and a sphere of radius $R_\alpha = 1.3 A_\alpha^{1/3}$ fm. Consequently, $r_0$ and $L$ are not considered as free parameters. The reciprocal moment of inertia $C = \hbar^2 / 2\mathcal{I}(r_0, L, h)$ can also be extracted from the soliton geometry. The energy density profile of the $\alpha$-cluster, considered as a quantum extended particle, characterized by the half-width associated with this energy density, $L_{en}$, is presented in Fig. 3 together with the corresponding solitonic solution. We remark that the potential energy, eq.(6), forms a valley which localises the soliton. The total energy density consists in a small maximum localised in this valley which characterises the associated quantum extended particle. The parameter $\beta$ and the range of the quantum number $k$ are fixed by the equation $L_{en} = L$. Under these restrictions we obtain the following values for the parameters: $r_0 = 0.62$, $L = 0.636$, $\beta = 0.345$ and $k = 0, 1$. The corresponding values for the soliton excitation energy are $\hbar\omega_k = 0.533$ and 1.0655 MeV for $k = 0, 1$ , correspondingly. We note that for other cluster configurations (i.e. larger mass of the colliding particle or of the target) a larger number of solitonic bands ( $\omega_k$'s), $k > 1$, is allowed. In this case we must take into account that only the $k = 0, 1$ solitonic modes are excited (the other two modes modes introduce negative $\omega^2$). For the perturbation in the third order we take only its highest term, i.e. that one corresponding to the highest value of $\omega_k$, $k = 0$. The total spectrum is limited to the following equation:

$$E_{I,N_0,N_1} = E_0 + \hbar\tilde{\omega}_0\Big(N_0 + \frac{1}{2}\Big) - B_0\Big(N_0 + \frac{1}{2}\Big)^2 + \hbar\tilde{\omega}_1\Big(N_1 + \frac{1}{2}\Big) + CI(I+1) \qquad (49)$$

We stress that this equation for the energy levels, though looks similar with those obtained in different other theoretical models [5-23], it is obtained from an unitary collective nonlinear model and has no connection with the interpretation of the previous ones. More than this, though the third order correction is obtained by comparison with the experiment, we note that we obtain its negative sign from the theory only.

The solitonic model, based on this nuclear molecular behaviour, succeeds to reduce the number of the parameters to only four: $L$, $E_0$, $h$ and $B_0$. The other parameters occuring in the theory (i.e. $\beta$, $r_0$ or equivalently $C$ and $\omega_k$) are obtained from the geometry of the system.

We compare our calculations with the experimental data given in [4], i.e. the excitation function for elastic scattering of alpha particles from $^{20}$Ne. By fitting the theoretical spectrum (49) with the above mentioned experimental resonances, we obtain, for the above theoretically calculated values of $\tilde{\omega}_k$ and $E_0 = 9.465$ MeV, $h = 0.121 R_{^{20}Ne}$, $C = 0.13052$ and $B_0 = 0.07878$ MeV a good agreement, shown in Figs. 4. The theoretical spectrum is ploted in the right part together with the values of the quantum numbers $(N_1, N_0)$, for even and odd values of $I$ (Fig.4a and 4b). The nonlinear degree of freedom which is related to the soliton quantum number $N_0$ is essential to reproduce the appearance of irregulary spaced levels with the same spin $I$. Reasonable good agreement is obtained and the solitonic description can describe the observed experimental energies and spins of the intermediate states of $\alpha +$ $^{20}$Ne system, like in the previously case for $\alpha$+$^{28}$Si system [28]. Both even and odd parities are fitted by using the same values for the parameters. The even and odd states form again mixed parity bands which can be interpreted like the rotating mass has an asymmetric shape as it is normal for the $\alpha$+$^{20}$Ne-system. The present theoretical description could imply that the alpha cluster as a soliton could be viewed as a new kind of nuclear deformation, with quasi-bound states which could be regarded as soliton and breather states.

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**Figure captions**

**Fig. 1** The classification of the solutions of the Lagrangean eq.(3) according to their time behaviour: a) instable soliton solution obtained for $x > 1$, $y < 1$, b) instable breather solution obtained for $x > 1$, $y > 1$, c) oscillating soliton solution for $x < 1$, $y > 1$ and d) breather solution for $x < 1$, $y < 1$. The corresponding examples are ploted against the $\phi$ coordinate (projected on the line) for three different moments of time. The decaying solutions (a,b) give informations about the life-time of the corresponding structures.

**Fig. 2** A pictorial view of the breather states on the nuclear surface presented for three quantum numbers $k = 0, 1, 2$. All three solutions have the same envelope of a solitonic type.

**Fig. 3** The shape of the static solution given in eq.(28) ("u"), its corresponding hydrodynamical soliton solution ("Soliton") and the energy density ("E[u]") are ploted against the $\phi$ coordinate projected on the line. The energy density reads a potential energy valley ("V[u]") in which is self-confined the soliton. The half-width L of the solution u and the corresponding half-width $L_{en}$ of the energy density profile are drawn with arrows, for comparison.

**Fig.4** The experimental and theoretical spectra of the excitation energy for $\alpha + ^{20}$Ne elastic scattering, for even (Fig. 4a) and odd (Fig. 4b) spins I. The experimental lines are divided in three categories: full lines, where the value of I is assigned with a good experimental precision and the reduced widths of the resonances are semnificative, dotted lines, where the experimental spin is not accurate obtained and dashed-dotted lines where the corresponding reduced widths of the resonances are smaller than 10, [4].