Analysis and classification of nonlinear dispersive evolution equations in the potential representation

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Abstract

For the subset of travelling solutions the potential representation of nonlinear dispersive evolution equations is introduced. It is a reduction of the third order partial differential equation to a first order ordinary differential equation. In this representation it can be shown that solitons and solutions with compact support do only exist in systems with linear or quadratic dispersion, respectively. In particular, this article deals with so called $K(n,m)$ equations. It is shown that these equations can be classified via a simple point transformation. As a result, all equations allowing for soliton solutions join the same equivalence class with the Korteweg-deVries equation being its representant.

1 Introduction

An increasing precision of measurements or the attempt of describing physical systems under extreme conditions as e.g. large amplitude excitations require to go beyond the linear limit. Neglecting dissipative effects in a first idealization, one encounters nonlinear dispersive partial differential equations describing the dynamical behaviour of the system – that is, in particular, one deals with third- and first-order spatial derivatives of the wave function u or powers of u, i.e. $(u^m)_{xxx}$, describing (nonlinear) dispersion, and $(u^n)_x$, and a dynamical term, i.e. the time derivative of the wave function, u_t.

A special type of nonlinear dispersive evolution equations in $1+1$ dimensions are the so-called $K(n,m)$ equations, $(u^m)_{xxx} - A(u^n)_x + u_t = 0$. The most prominent examples for these equations are the Korteweg-deVries (KdV) equation, $K(2,1)$, or the modified KdV equation, $K(3,1)$ []. These equations are of particular interest because they are able to describe the motion of stable localized solitary waves (solitons) that are observed in a vast variety of physical systems and became attractive for important applications as, e.g., optical bits for the data transfer in glass fibers [] or as an explanation of, e.g., cluster radioactivity [].

In particular the considerations in this article will be carried out for the specific case of travelling solutions. It is shown that this restriction allows a special representation of the nonlinear dispersive evolution equations, which is called the potential representation. It resembles an energy conservation law containing a nonrelativistic kinetic energy term and a potential energy []. The potential representation, being a first order ordinary differential equation, constitutes an enormous simplification of the original problem. Gross properties of the solutions can be read off directly from the potential function without the actual need to solve the differential equation. The conditions for solitary waves and solitons can thus be easily stated qualitatively. Moreover, the investigation of the potential picture directly reveals that solitons can only emerge in systems with linear dispersion. Compactons, i.e. solitary waves with compact support, only exist in systems with quadratic dispersion. These results are discussed in more detail for the specific cases of systems that are modeled by $K(n,m)$ equations (see e.g. []).

In restricting the considerations to a particular subset of solutions a $K(n,m)$ equation can be transformed into another $K(N,M)$ equation by a simple point transformation. This point transformation defines an equivalence relation between $K(n,m)$ equations and divides the infinite set of $K(n,m)$ equations into equivalence classes of connected equations.

2 Potential Representation

2.1 Reformulation of the nonlinear evolution equation

We consider a general nonlinear dispersive evolution equation of an autonomous one-dimensional non-dissipative dynamical system

$$(u^m)_{xxx} = V(u)u_x - u_t \quad , \tag{1}$$

where $V(u)$ is an arbitrary integrable and continuous function of u. The subscripts denote the partial differentiation with respect to the index. We focus on travelling solutions with the space-time dependence $u(x,t) = u(x-vt) = u(\xi)$, with v denoting the speed, so that the partial differential equation (**??**) is reduced to the ordinary differential equation

$$(u^m)_{\xi\xi\xi} = V(u)u_\xi + vu_\xi = \mathcal{V}(u)u_\xi \quad , \tag{2}$$

Equation (**??**) can be integrated once

$$(u^m)_{\xi\xi} = \int_0^u \mathrm{d}t\mathcal{V}(t) - C_1 \quad , \tag{3}$$