

Wavelets and Quantum Algebras

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Abstract:

A non-linear deformed associative algebra is realised in terms of translation and dilation operators, and a wavelet structure generating algebra is obtained. We show that this algebra is a q -deformation of the Fourier series generating algebra, and reduces to this for certain value of the deformation parameter. This algebra is also homeomorphic with the q -deformed $su_q(2)$ algebra. This is achieved by considering a functional that maps the q -algebra $su_q(2)$ to the deformed algebra.

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In order to give an algebraic approach (definition) to multiresolution analysis [2] in an algebraic framework, one needs to modify the traditional operations, like differentiation or Lie algebras, in a suitable way. The basic tools of the multiresolution analysis, i.e. the scaling and wavelet functions are in general only continuous functions. Consequently one needs the introduction of the q -deformed derivatives which can be applied to such non-differentiable functions. Such operators are involved together with q -algebras.

Quantized universal enveloping algebras, also called q -algebras, refer to some specific deformations of (the universal enveloping algebra of) Lie algebras, to which they reduce when the deformation parameter q is set equal to one [3]. The simplest example of q -algebra, $su_q(2) \equiv U_q(\mathfrak{su}(2))$, was first introduced by Sklyanin [4], and independently by Kulish and Reshetikhin [5] in their work on Yang-Baxter equations. A Jordan-Schwinger realization of $su_q(2)$ in terms of q -bosonic operators was then derived by Biedenharn [6] and Macfarlane [7]. Since then, $su_q(2)$ has been applied in various branches of physics. It has been found suitable, for instance, for the solution of deformed spin-chain models [8], as well as for the approximate description of rotational spectra of deformed nuclei [9]. In addition to the usual version of the deformed $su(2)$ algebra, namely $su_q(2)$, several generalized forms of the algebra have been introduced. Deformations involving one arbitrary function of J_0 were independently proposed by Polychronakos [10], Roček [11] and Ludu [23]. Their representation theory is characterized by a rich variety of phenomena, which might be of interest in applications to particle physics. Deformations of $su(2)$ involving two arbitrary functions of J_0 were introduced by Delbecq and Quesne [13]. Contrary to the former deformations, for which the spectrum of J_0 is linear as for $su(2)$, the latter give rise to exponential spectra. Such spectra did recently arouse much interest in various contexts, for instance in connection with alternative Hamiltonian quantization [14], exactly solvable potentials [15], q -deformed supersymmetric quantum mechanics [16], and q -deformed interacting boson model [17]. As MRA is essentially based on wavelets which analyse the signal at different scales, the magnitude of these scales being discretized by powers of 2 [1], this typical peculiarity of these q -deformed algebras (the exponential spectrum of J_0) is the most appropriate algebraic frame for the MRA analysis.

The q -deformed algebras could be realised in terms of q -differential operators [7]. It is this approach which we shall use in our investigations.

In the present letter, we shall address the problem of obtaining a q -deformed algebra,

realised in terms of q -differential operators (i.e. translation and dilation), which should fulfill the algebraic frame for the wavelets. This means that the scaling equation, which provides the mother scaling function, and the mother wavelet, should be obtained as an eigenvalue problem from this closed algebra. This algebra should also have two features: on one side it should be deformable towards the Fourier series generating algebra (and in this way we can regard wavelets as q -deformed generalisations of the Fourier series), and on the other side, it should reduce to $su_q(2)$ algebra, in order to benefit of all the properties of this well investigated q -deformed algebra (its unirreps and Casimir operator). For such purposes, we shall use a variant and extension of the deforming functional technique, wherein we obtained functionals mapping the generators of a nonlinear algebra both to those of the Lie algebra $M(2)$ of translations and dilations of the \mathbf{R}^2 plane (which generates the Fourier complex exponential) and to the $su_q(2)$ one. By considering the special case where the $s \rightarrow s_0$ this algebra reduces to $su_q(2)$.

Different from the ordinary Fourier transform, which reproduces a function as a superposition of complex exponentials, or from the windowed Fourier transform, which introduces a scale into the analysis of signals (the width of the window), the Multi Resolution Analysis (MRA) [2], processes the signal locally, without prejudice the scale. Wavelet analysis is precisely a scale-independent method. In the algebraic picture the Fourier Transform could be realised in different ways. A possibility is to use a Lie algebra generated by the three generators: $J_0 = -i\partial$, $J_+ = e^{ix}$, $J_- = (J_+)^\dagger = e^{-ix}$ having the commutators: $[J_0, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 0$, which is isomorph to $M(2)$. The space of the representations is generated by the complex exponentials $|n\rangle = e^{inx}$, $J_0|n\rangle = n|n\rangle$, $J_\pm|n\rangle = |n \pm 1\rangle$. The generators J_\pm act like ladder operators on the $|n\rangle$ states by increasing/decreasing the scale (i.e. n) with one integer. In order to obtain a scale-invariant algebra we have to introduce other generators and representations, which fulfill different commutators, which allow us to obtain the scale-invariant equation. Consequently, some of the generators of this new algebra should satisfy invariant-like equation, of the form $J_\pm|\phi\rangle = |\phi\rangle$. The corresponding q -deformed ladder operators will not change the scale anymore, but keep the scale-invariance due to this equation. Of course all the corresponding algebraic analysis should be based only on the mother function $|\phi\rangle$. The MRA has a typical feature: the scale of different analysing functions has discrete values, not equidistant like in the FT case, but having exponential behaviour, in terms of powers of 2. The q -deformed algebras allow the existence of such

exponential spectra [..]. In fact this is one of their most important modification with respect to the traditional undeformed, Lie algebras. This observation shows us that, finding the most appropriate q-deformed algebra for a given wavelet analysis is the natural algebraic frame-work for MRA.

The wavelets are generated by $L_2(\mathbf{R})$ integrable functions, and there is no need to introduce smoother functions like differentiable functions of order one. This is an important freedom of the wavelet analysis (one can work with step-like functions) but, on the other side, it generates a problem: there is no more advantage of working with derivatives, because they are no more defined in this context. The q-deformed algebraic approach restaures this advantage of the traditional FT, by introducing the q-deformed derivative. The action of this q-derivative, $[\partial]$, on continuous functions, has sense because it acts as finite difference operators. More important than this, the q-derivative reduces to the normal derivative when the deformation parameter s goes to 0. We have then a usefull tool, which provide the necessary finite-difference frame of work with wavelets, and the connection with the undeformed object. We note also that the q-derivative gives us a continuous transformation of the function on which is applied, towards its derivative, through the continuous variation of the deformation parameter.

In the following we work on the space of compact support $L_2(\mathbf{R})$ real functions $f(x)$. We introduce in this space the operators: $T^\alpha = e^{\alpha\partial}$, and $D^\beta = e^{\beta \ln 2x\partial}$, i.e. translation and dilation operators with the corresponding action: $T^\alpha f(x) = f(x + \alpha)$, $D^\beta f(x) = f(2^\beta x)$. Here α and β are real numbers and we denoted by ∂ the differentiation with respect to x . These operators have the properties: $(T^\alpha)^\dagger = T^{-\alpha}$, $(D^\beta)^\dagger = D^{-\beta}$ and they could be re-scaled in order to be unitarian operators. We have also the relation $T^\alpha D^\beta = D^\beta T^{2^\beta + \alpha}$. We stress that we define the operators T and D only from their action on $L^2(\mathbf{R})$ functions. If these operators act on $C^\infty(\mathbf{R})$ functions we can write explicetly their exponential form as formal series of derivative operator. More general, we can work with formal functions of T , $f(T)$, when these function are well define on $L^2(\mathbf{R})$. The best example is given in the case when f is an integer holomorphic function, and the action is taken on the compact supported function subspace of $L^2(\mathbf{R})$. In this last case, the action of $f(T) = \sum_{k \in Z} C_k T^k$ reduces to the action of the corespondently Laurent polynomial, i.e. if $supp\Phi(x) \in (-M, M)$ then $f(T)\Phi = \sum_{k=-2M}^{2M} C_k T^k \Phi$.

In the following we shall denote simply $T^1 = T$, $D^1 = D$. We introduce the q-

deformation of the object x (a number, a function or an operator) in the standard form [2-7], for real deformation parameter $q = e^s$, $s \in \mathbf{R}$:

$$[x]_s = [x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh(sx)}{\sinh s}, \quad (1)$$

which tends to x when $s \rightarrow 0$. Consequently we define the q -deformation of the derivative in the form:

$$[\partial]f(x) = \frac{e^{s\partial} - e^{-s\partial}}{\sinh s} f(x) = \frac{f(x+s) - f(x-s)}{\sinh s}, \quad (2)$$

and:

$$[x\partial]f(x) = \frac{f(2^s x) - f(2^{-s} x)}{\sinh s}. \quad (3)$$

When $s \rightarrow 0$ $[\partial] \rightarrow \partial$ and $[x\partial] \rightarrow x\partial$. With eqs.(..) we have the relation between the q -deformed derivatives and the translation and dilation operators:

In order to realise a non-linear algebra for the wavelets generation, we introduce the operators:

$$J_0 = \frac{e^{s\alpha\partial} - e^{-s\alpha\partial} \cos s\pi}{2 \sinh s \xi(\alpha)} \quad (4)$$

where $s = \ln(q)$ is the real q -deformation parameter, $\xi(\alpha)$ is a function of class $C^1(\mathbf{R})$ having the property: $\xi(0) = 0$, $\xi'(0) = 1$ and $\alpha \in \mathbf{R}$ is a parameter which fixes the necessary form for the J_0 operator in different limiting cases (different fixed values for s). In the limits $s \rightarrow 0, \frac{1}{2}, 1$ and 2 we have:

$$J_0(0) = \frac{\alpha\partial}{\xi(\alpha)}, \quad (5)$$

$$J_0\left(\frac{1}{2}\right) = \frac{1}{2 \sinh(1/2) \xi(\alpha)} T^{\frac{\alpha}{2}}, \quad (6)$$

$$J_0(1) = \frac{T^\alpha + T^{-\alpha}}{2 \sinh(1) \xi(\alpha)}, \quad (7)$$

$$J_0(2) = \frac{T^{2\alpha} - T^{-2\alpha}}{2 \sinh(2) \xi(\alpha)}. \quad (8)$$

The general form eq.(1) gives different possibilities of approach for the operator $J_0(s)$. In the limit $s \rightarrow 0$ J_0 becomes the normal derivative with respect to x , eq.(..). In the limit $s = \frac{1}{2}$, we obtain the simplest form for J_0 , i.e. a power of the (unitary) translation operator T , eq.(..). This limit, together with the limiting form given in eq.(..) are useful when we construct the homomorphism of the $\mathcal{A}_{T,D}$ algebra with $su_q(2)$ q -deformed algebra. Eq.(..) gives a self-adjoint operator, $J_0(1) = J_0(1)^\dagger$, which is a sort of q -deformation of the unity,

i.e. $J_0(1) \rightarrow \frac{1}{\sinh(\pi)}I$ when $\alpha \rightarrow 0$. The last example, eq.(.), gives an anti-unitary operator which represents the q-derivative with respect to the α -deformation, for $\xi(\alpha) = \frac{\sinh(\alpha)}{\sinh(2)}$:

$$J_0(2) = [2\partial]_\alpha, \quad (9)$$

and we have $J_0(2) \rightarrow 2\partial$ in the limit $\alpha \rightarrow 0$. We introduce other two operators:

$$J_\pm(s) = \frac{1}{2}e^{\mp ix \frac{s-1}{2}} D^{\mp s} e^{\mp ix \frac{s-1}{2}} (1 + T^{\pm s}), \quad (10)$$

having the property $(J_+)^{\dagger} = J_-$. In the limit $s \rightarrow 0$ we obtain:

$$J_\pm(0) = e^{\pm ix}. \quad (11)$$

By choosing $\xi(\alpha) = i\alpha$, we obtain that $\mathcal{A}_{T,D}$ is homeomorphic with the Lie algebra $M(2)$ of translations and dilations in the \mathbf{R}^2 plane, i.e. the Fourier series generating algebra, defined by the relations:

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 0 \quad (12)$$

The unitary irreducible representations (unirreps) of this Lie algebra are based, like in the case of $su(2)$, on the eigenvectors of the self-adjoint operator J_0 , $J_0|a \rangle = a|a \rangle$. Suppose we know two distinct eigenstates, $|a \rangle$ and $|a' \rangle$, by using eq.(.), we obtain

$$\langle a|[J_0, J_+]|a' \rangle = \langle a|J_+|a' \rangle \quad (13)$$

and from here: $a' = a - 1$. Hence all the eigenvalues of J_0 are integer multiples of the smallest, non-zero one, and the spectrum of J_0 is equidistant. The Casimir operator for this algebra is given by J_+J_- . A basis of unirreps consists in the Fourier basis $e^{inx}_{n \in \mathbf{Z}}$ with the realisation $J_0 = -i\partial$, $J_\pm = e^{\pm ix}$. The operators J_\pm change, by their action on the basis functions, the scale of these functions, i.e. $n \rightarrow n \pm 1$.

By using eqs.(4,10) we obtain the commutator relations:

$$[J_0, J_+] = G_s(T)J_+, \quad (14)$$

$$[J_0, J_-] = -J_-G_s(T), \quad (15)$$

$$[J_+, J_-] = H_s(T) \quad (16)$$

with:

$$G_s(T) = J_0(T) - \frac{T^{2^s s \alpha} e^{is\alpha(1+2^s)\frac{s-1}{2}} - T^{-2^s s \alpha} e^{-is\alpha(1+2^s)\frac{s-1}{2}} \cos s\pi}{2 \sinh s\xi(\alpha)} \quad (17)$$

$$F_s(T) = \frac{1}{4} \left((1 + e^{is(1+2^s)\frac{s-1}{2}} T^{2^s s})(1 + T^{-s}) - (1 + e^{is(1+2^{-s})\frac{s-1}{2}} T^{-2^{-s} s})(1 + T^s) \right) \quad (18)$$

These relations define a non-linear associative algebra, depending on the functions $G_s(T)$ and $F_s(T)$, similar with the algebra introduced in [23].

More general, we can choose the following generators:

$$J_0 = g(T) \quad (19)$$

and

$$J_{\pm} = e^{\mp ix \frac{s-1}{2}} D^{\mp s} e^{\mp ix \frac{s-1}{2}} h(T^{\pm}, s) \quad (20)$$

where the operator functions $h(T, s)$ and $g(T, s)$ are integer functions of T , holomorphic in a neighborhood of 1, $(h(T, s))^{\dagger} = h(T^{-1}, s)$ and depend on s such that in the limit $s \rightarrow 0$ we have $g(T, 0) \rightarrow -i\partial$, $h(T, 0) \rightarrow 1$. We have then the commutator relations between these generators:

$$[J_0, J_+] = G(T, s)J_+, \quad (21)$$

$$[J_0, J_-] = -J_-G(T, s), \quad (22)$$

$$[J_+, J_-] = H(T, T^{-1}, s) \quad (23)$$

with:

$$G(T, s) = g(T, s) - g\left(T^{2^s} e^{i(1+2^s)\frac{s-1}{2}}, s\right) \quad (24)$$

$$H(T, s) = h\left(e^{i\frac{s-1}{2}(1+2^s)} T^{2^s}, s\right)h(T^{-1}, s) - h\left(T^{-2^{-s}} e^{i\frac{s-1}{2}(1+2^{-s})}, s\right)h(T, s) \quad (25)$$

we note that the first two commutators of the algebra $\mathcal{A}_{g,h}$ do not depend on the function $h(T, s)$, and that the third commutator does not depend on the function $g(T, s)$. Consequently, in some sense, the functions g and h are decoupled and could be arranged in a convenient way for different purposes.

In order to construct a complete non-linear algebra for wavelets of any order, we give the general form for the operator $h(T, s)$ as a polynomial of order M , with coefficients being smooth functions of the q-deformation parameter s :

$$h(T, s) = \sum_{k=0}^M C_k(s) T^{k\alpha(s)}, \quad (26)$$

where the coefficients must satisfy the constrains:

$$\sum_{k=0}^M C_k(s) = 2 \quad (27)$$

$$\sum_{n=0}^{M-2k} C_n(s)C_{n+2k}(s) = \frac{1}{2}\delta_{k,0} , \quad k = 0, 1, \dots, N - 1, \quad (28)$$

$$C_k(-s) = (-1)^k C_k(s) \quad \alpha(-s) = -\alpha(s) + \lambda, \quad (29)$$

with λ an odd integer, $M = 2N - 1$, for any s . The condition in eqs.(..) provide the necessary restrictions for the scaling function $\Phi(x)$. Eq.(..) provides the compatibility of $h(T, s)$ with the scaling equation (the averaging property). If we integrate the scaling equation $D^{-1}\Phi = h(T, s)\Phi$, over \mathbf{R} , we have to...

This is easy to check because we have from eq.(..0

$$\begin{aligned} \delta_{k,0} = \langle \Phi_k, \Phi \rangle &= 2 \langle D^{-1}\Phi_k, D^{-1}\Phi \rangle + 2 \langle h(T, s)\Phi_{2k}, h(t, s)\Phi \rangle \\ &= 2 \sum_{j,k=0}^M C_j C_k \langle \Phi_{2k+j}, \Phi_n \rangle = 2 \sum_{j,k=0}^M C_j C_{2k+j} \end{aligned} \quad (30)$$

This condition, eq.(..), is valid only for M odd (otherwise it results in $M = 0$ which contradicts the definition eq.(..)) and consequently consists in $k = 0, 1, \dots, [M/2] = N - 1$ quadratic algebraic equations for the functions $C_k(s)$. If we choose s_1 such that $\alpha(s_1) = 1$ the operator $h(T, s_1)$ generates the scaling equation. More than this, for $s = -s_1$ we have, through eq.(..),:

$$h(T, -s_1) = T^\lambda h(-T^{-1}, s) = g(T, s) \quad (31)$$

which is the operator which generates the wavelet function, i.e. $g(T, s)\Phi = \Psi$.

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