P43F-3335

Jacobi Integral
A real-world application of the Lagrangian formulation

J. A. Riousset[1], (jeremy.riousset@erau.edu) & M. Lingam[2]

[1]SAIL laboratory, PS Department, Embry-Riddle Aeronautical University, Daytona Beach, FL; [2]APSS Department, Florida Institute of Technology, Melbourne, FL

Abstract

The applicability of advanced classical mechanics (viz., the Lagrangian and/or Hamiltonian approaches) to real-world problems may not always seem straightforward, despite the mathematical rigor and elegance of this field. Here, we present a proof of the Jacobi integral using the Lagrangian formulation as a viable alternative to the usual demonstration using Newton's second law. The result represents a useful example of how advanced classical mechanics can provide a significant advantage over standard methods (i.e., Newton's laws). We conclude with an illustration of the Jacobi integral in our Solar system: the Pluto-Charon system.

I. Introduction

Figure 1: This composite of enhanced color images of Pluto (lower right) and Charon (upper left), was taken by NASA's New Horizons spacecraft as it passed through the Pluto system on July 14, 2015 (Image Credit: NASA/JHUAPL/SwRI).

- CR3BP: Standard problem in classical mechanics [e.g., 1–3].
- Mostly Newtonian approaches for treatments of Jacobi integral [e.g., 4–10].
- Need for a self-contained derivation of Jacobi integral via Lagrangian mechanics.
- Application to the Pluto-Charon system with SI units over normalized units.
- Comparison with Earth-Moon system.
- Orbital mechanics as a potent example of advanced classical mechanics to simplify a real-world problem.

Object	Symbol	Mass (km)	Radius (km)	Distance Gm_i (km)
Earth	E ⊕	5.97×10^{24}	6,378.0	4,670.7
Moon	M ☾	7.35×10^{22}	1,737.0	379,729.3
Pluto	P ♇	1.31×10^{22}	1,188.3	2,122.4
Charon	C	1.59×10^{21}	606.0	17,518.0

Table 1: Data sheet for the Earth-Moon & Pluto-Charon systems.

II. Setup

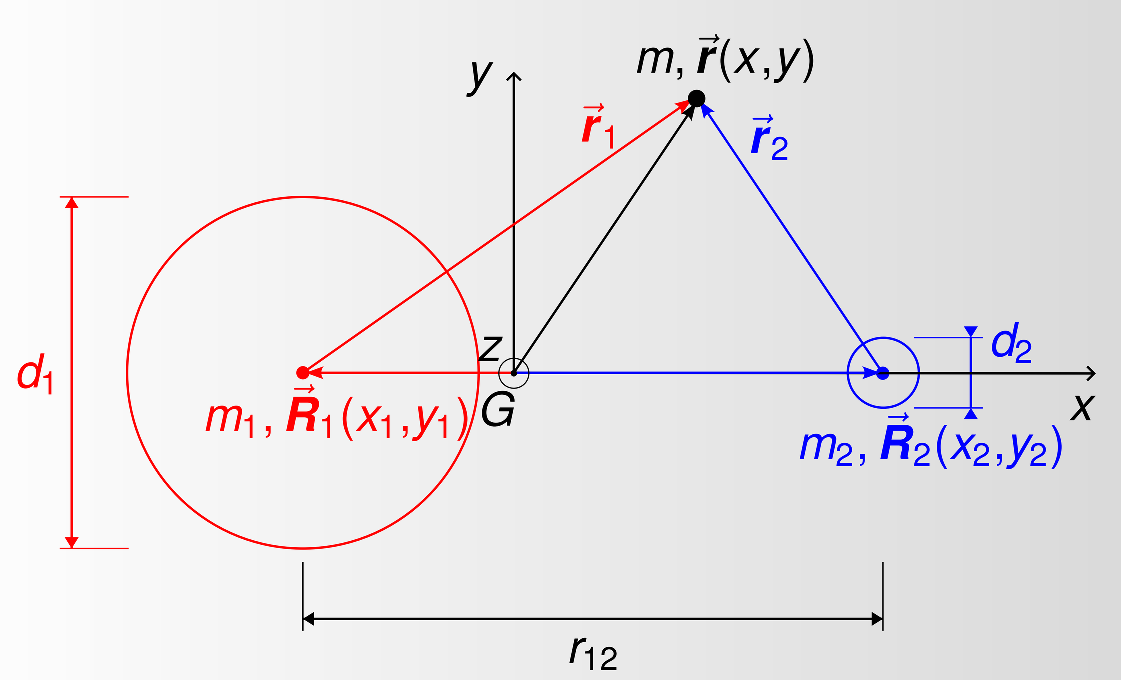

Figure 2: The Circular Restricted 3-Body Problem (CR3BP). Two point-masses m_1 and m_2 are located at the center of two spheres of diameters d_1 and d_2, placed at $\vec{R}_1$ and $\vec{R}_2$. They are at a distance of r_{12} from each other. The third mass m is located at $\vec{r}$ from the center of mass G, and at $\vec{r}_1$ and $\vec{r}_2$ from m_1 and m_2, respectively.

Invariants

$$\vec{v}_G = \vec{0} \qquad r_{12} = \left|\vec{R}_2 - \vec{R}_1\right| \qquad \vec{\Omega} = \frac{2\pi}{P}\hat{e}_z \qquad P = \sqrt{\frac{4\pi^2}{\mu} r_{12}^3}$$

Variables

$$\vec{r} = x\hat{e}_x + y\hat{e}_y \qquad \vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y \qquad \vec{a} = \ddot{x}\hat{e}_x + \ddot{y}\hat{e}_y$$

III. Lagrange's Equations

Specific Kinetic Energy

$$\vec{v}_{\text{abs}} = \cancelto{\vec{0}}{\vec{v}_G} + \vec{v} + \vec{\Omega} \times \vec{r} = (\dot{x} - \Omega y)\hat{e}_x + (\dot{y} + \Omega x)\hat{e}_y$$

$$T = \frac{\vec{v}_{\text{abs}}^2}{2} = \frac{(\dot{x} - \Omega y)^2 + (\dot{y} + \Omega x)^2}{2}$$

Gravitational Potential

$$V = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}$$

Lagrangian

$$L = T - V = \frac{(\dot{x} - \Omega y)^2 + (\dot{y} + \Omega x)^2}{2} + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

Lagrange's Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \quad \text{where} \begin{cases} q_1 = x \;;\; \dot{q}_1 = \dot{x} \\ q_2 = y \;;\; \dot{q}_2 = \dot{y} \end{cases}$$

Using Cartesian coordinates,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_1} = \ddot{x} - \Omega\dot{y} \qquad \frac{\partial L}{\partial q_1} = \Omega\dot{y} + \Omega^2 x - \frac{\mu_1(x - x_1)}{r_1^3} - \frac{\mu_2(x - x_2)}{r_2^3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_2} = \ddot{y} + \Omega\dot{x} \qquad \frac{\partial L}{\partial q_2} = -\Omega\dot{x} + \Omega^2 y - \frac{\mu_1(y - y_1)}{r_1^3} - \frac{\mu_2(y - y_2)}{r_2^3}$$

yield the classical equations of motion:

$$\ddot{x} = 2\Omega\dot{y} + \Omega^2 x - \frac{\mu_1(x - x_1)}{r_1^3} - \frac{\mu_2(x - x_2)}{r_2^3},$$

$$\ddot{y} = -2\Omega\dot{x} + \Omega^2 y - \frac{\mu_1(y - y_1)}{r_1^3} - \frac{\mu_2(y - y_2)}{r_2^3}.$$

Calculate $\dot{x}\ddot{x} + \dot{y}\ddot{y}$ & simplify:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Omega^2\left(x^2 + y^2\right)}{2} + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} - \frac{\dot{x}^2 + \dot{y}^2}{2}\right)$$

So ultimately:

$$\boxed{C_J = \Omega^2 r^2 + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} - v^2}.$$

IV. Results & Discussion

Pseudo-Potential

$$U = C_J + v^2 = \Omega^2 r^2 + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2}.$$

Zero-Velocity Curves

$$U(v = 0) = C_J = \Omega^2 r^2 + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2}$$

$$\frac{\mathcal{E}}{m} = -\frac{1}{2}C_J = \underbrace{\left(\frac{1}{2}v^2 - \frac{1}{2}\Omega^2 r^2\right)}_{\text{(I)}} + \underbrace{\left(-\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}\right)}_{\text{(II)}}$$

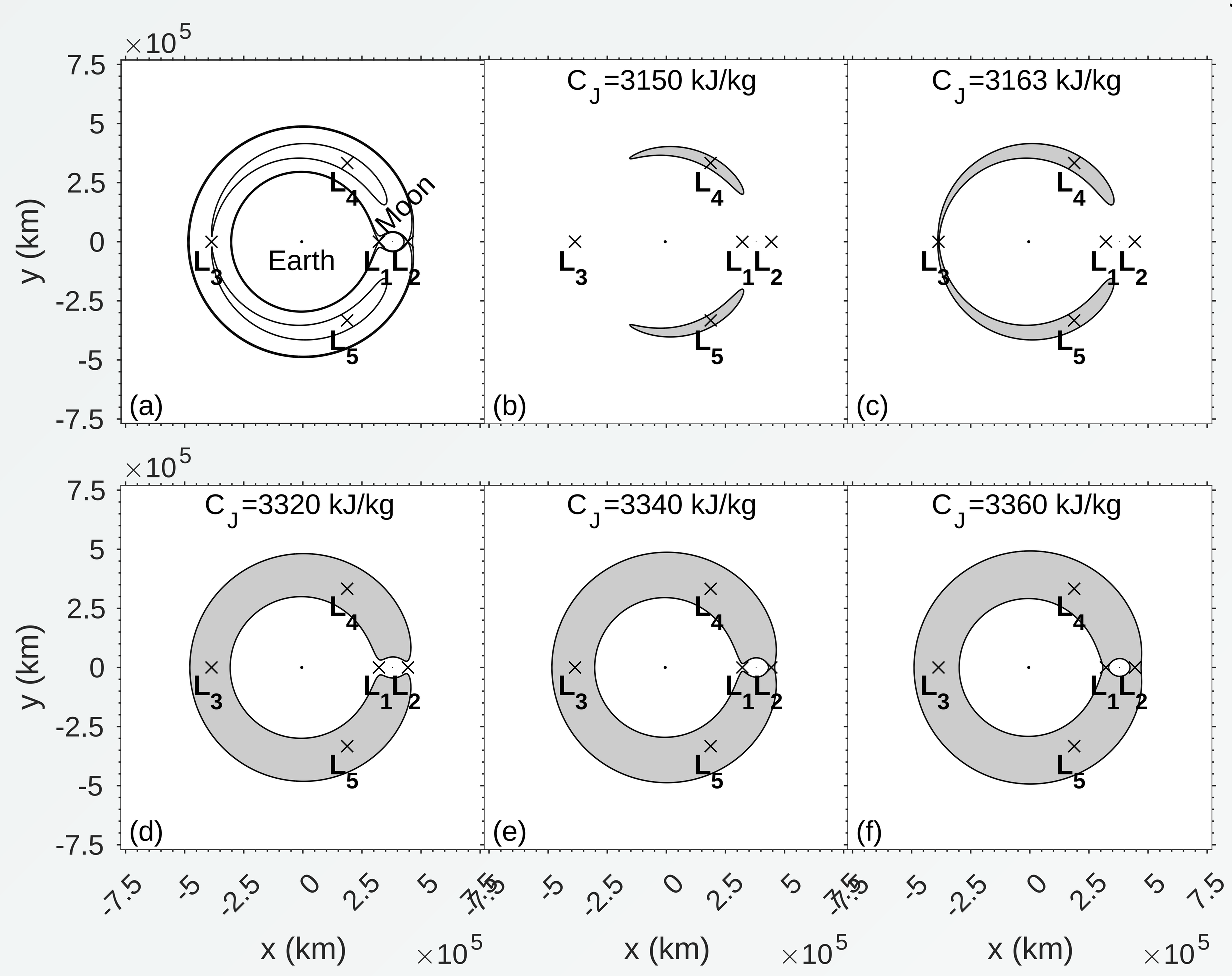

Figure 4: Zero-velocity surfaces for the Earth-Moon system for multiple C_J constant.

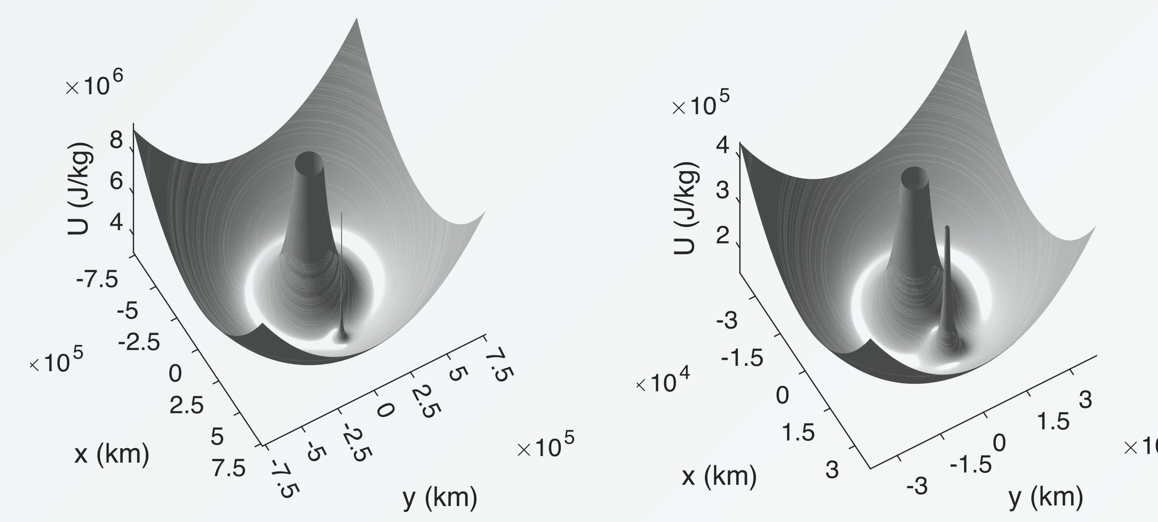

Figure 3: Pseudo potential for the Earth-Moon (left) and Pluto-Charon (right) systems: $U = C_J + v^2$ vs. x-y (see Figure 2 and [11]).

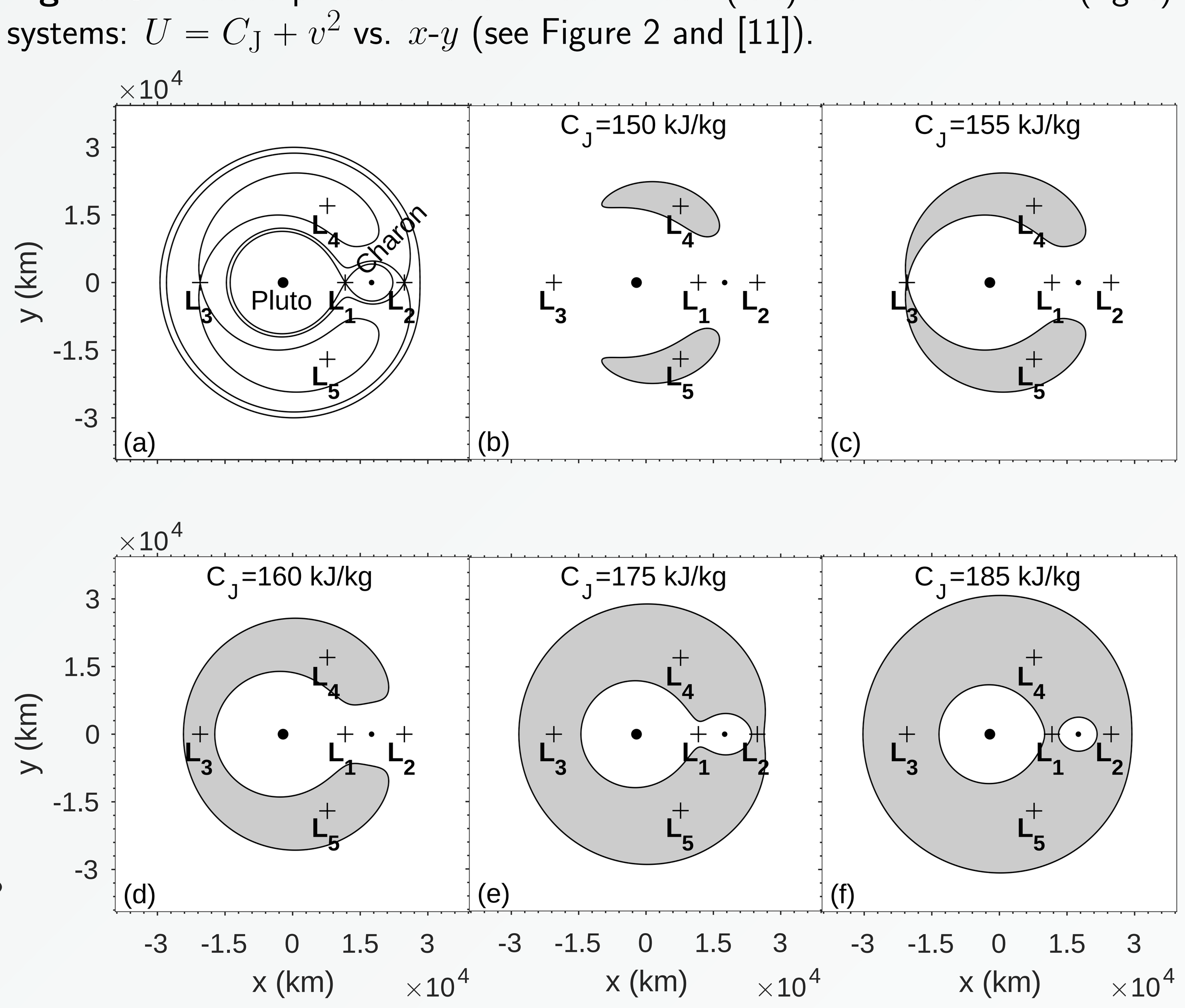

Figure 5: Same as Figure 4 for the Pluto-Charon system.

References

[1] J. Barrow-Green, *Poincaré and the three body problem*, AMS, Providence, RI, 1997.
[2] M. C. Gutzwiller, Rev. Mod. Phys. **70**, 589 (1998).
[3] M. Valtonen et al., *The Three-body Problem from Pythagoras to Hawking*, Springer International Publishing, Cham, CH, 2016.
[4] C. D. Murray and S. F. Dermott, *Solar System Dynamics*, Cambridge Univ. Press, Cambridge, UK, 1999.
[5] R. H. Battin, *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA, Reston, VA, revised edition edition, 1999.
[6] D. A. Vallado and J. Wertz, *Fundamentals of Astrodynamics and Applications*, Space Technology Library, Microcosm Press, Hawthorne, CA, fourth edition, 2013.
[7] I. de Pater and J. J. Lissauer, *Planetary Sciences*, Cambridge Univ. Press, Cambridge, UK; New York, NY, updated second edition, 2015.
[8] K. F. Wakker, *Fundamentals of Astrodynamics*, Institutional Repository Library, Delft, NL, 2015.
[9] P. Gurfil and P. K. Seidelmann, *Celestial Mechanics and Astrodynamics: Theory and Practice*, volume 436 of *Astrophysics and Space Science Library*, Springer-Verlag, Berlin, DE, 2016.
[10] H. Curtis, *Orbital Mechanics for Engineering Students*, Butterworth-Heinemann, Amsterdam, NL, revised fourth edition, 2020.
[11] MATLAB Monkey, CRTBP Pseudo-Potential and Lagrange Points, https://www.matlab-monkey.com/celestialMechanics/CRTBP/LagrangePoints/LagrangePoints.html, Accessed: 2023-05-16.

IV. Conclusions

In this work, we tackle the Jacobi constant from the Circular Restricted 3-Body Problem via advanced classical mechanics (the Lagrangian formulation), and thereafter provide a real-world example in the form of the Pluto–Charon system. The principal results and contributions from this work can be summarized as follows:

1. The Circular Restricted 3-Body Problem (CR3BP) offers a practical illustration of the benefits of the Lagrangian formalism over the classical approach entailing Newton's laws of motion.
2. The Pluto–Charon system offers ideal conditions to demonstrate a real-world instantiation of the CR3BP in our Solar system.
3. The quantitative results ensuing from our study are consistent with previous publications in the peer-reviewed literature, [e.g., Refs. 6; 7; 10].

Acknowledgements & CRediT

JAR: Conceptualization, Methodology, Software, Formal Analysis, Investigation, Resources, Data Curation, Writing - Original draft preparation, Visualization, Supervision, Project Administration, Funding Acquisition **ML:** Validation, Investigation, Writing - Original draft preparation.

We gratefully acknowledge Prof. Jean Carlos Perez at the Florida Institute of Technology for useful discussions of the Lagrangian formalism. This work was supported by the NSF under CAREER grant 2047863 to ERAU.